# Adaptive control of uncertain nonlinear systems

Ashman, J. A.

*Award date:*
1994

*Awarding institution:*
University of Bath

Link to publication

# Adaptive Control of Uncertain Nonlinear Systems

submitted by

J.A. Ashman

for the degree of Ph.D.

of the

University of Bath

1994



Signature of Author ............ J Ashman ............................................

J.A. Ashman

Thanks to Neal, Mum and Dad.

Thanks to Gene Ryan for all the help and encouragement.

# Contents

# Summary

Adaptive control of uncertain, nonlinear systems is considered. In much theory on control, the plant is assumed known and a control is designed using this plant information. By contrast, in this thesis exact information on the plant is not assumed; instead the plant is assumed only to belong to a class of plants on which only structural information is available. A control is constructed using this limited information.

We consider three control objectives. The first is the attractivity of the state origin. The final two objectives are those of an output asymptotically tracking or $\lambda$-tracking a given reference signal. Discontinuous feedbacks are constructed to achieve the first two control objectives. For analytical reasons these feedbacks are embedded in set-valued controls, and consequently the problem is formulated as a differential inclusion.

We study (a) a class of nonlinearly-perturbed linear systems and (b) a class of more general nonlinear systems. For the nonlinearly-perturbed linear systems, existing theory uses *unmixing sets*: for given $m$, this is a finite set of matrices $\{U_n\}_{n\in\{1,\ldots,N\}}$ such that for every real, invertible, $m \times m$ matrix $M$, there exists a $n \in \{1, \ldots, N\}$ such that $\sigma(MU_n) \subset \mathbb{C}_+$. For every $m$, existence of such a set is guaranteed, but the problem of determining a general practical method for its construction remains open. Aspects of this problem are addressed.

In the case (b) of uncertain systems with more general nonlinearities, we study systems in a normal form with a similar structure to the normal forms of the nonlinearly-perturbed linear systems. Three subclasses are considered, primarily classified by the type of stability a subsystem (the zero dynamics) possesses. Here, inverse Liapunov theory is prominent in the analysis. The cases of single-input and multi-inputs are considered. For the latter an unmixing set constructed earlier in the thesis is exploited for this new problem.

# Notation

$\mathbb{N}$ : the natural numbers.

$\mathbb{N}_0$ : the natural numbers and zero.

$\mathbb{R}$ : the real numbers.

$\mathbb{R}_+$ ($\mathbb{R}_-$) : the positive (negative) real numbers.

$\mathbb{C}$ : the complex numbers.

$\mathbb{C}_+$ ($\mathbb{C}_-$) : the open right (left) half complex plane, ie. the set $\{z \in \mathbb{C} \,|\, \mathrm{Re}(z) > 0\}$ ($\{z \in \mathbb{C} \,|\, \mathrm{Re}(z) < 0\}$).

$\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ : the Euclidean inner-product, ie. with $x := [x_1, x_2, \ldots, x_n]^T$, $y := [y_1, y_2, \ldots, y_n]^T \in \mathbb{R}^n$, $\langle x, y \rangle := \sum_{i=1}^n x_i y_i$.

$\| \cdot \| : \mathbb{R}^n \to \overline{\mathbb{R}}_+$ : the Euclidean-norm, ie. with $x \in \mathbb{R}^n$ as above, $\|x\| := \sqrt{\sum_{i=1}^n x_i^2}$.

$C^p(\mathbb{R}^n, \mathbb{R}^m)$ : the set of $p$-times continuously differentiable functions from $\mathbb{R}^n$ to $\mathbb{R}^m$. When the domain and codomain are implicit, we use $C^p$.

$W^{1,\infty}(\mathbb{R}, \mathbb{R}^n)$ : the set of functions from $\mathbb{R}$ to $\mathbb{R}^n$ which are absolutely continuous on compact intervals and which are bounded with essentially bounded derivative.

$L^p(\Omega; \mathbb{R}^n)$ : the set of $p$-integrable functions $\Omega \to \mathbb{R}^n$, where $\Omega$ is a Lebesgue measurable set.

$\mathcal{GL}(m; \mathbb{R})$ : the set of invertible, $m \times m$ matrices with real entries.

For the following, let $\epsilon > 0$, $A$, $B \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$.

$\overline{A}$ : the closure of the set $A$.

$\mathbb{B}_n$ : the open unit ball, ie. the set $\{z \in \mathbb{R}^n \mid \|z\| < 1\}$. When the dimension is implicit we shall use $\mathbb{B}$.

$\epsilon\mathbb{B}_n$ : the open ball of radius $\epsilon$, ie. the set $\{z \in \mathbb{R}^n \mid \|z\| < \epsilon\}$. When the dimension is implicit we shall use $\epsilon\mathbb{B}$.

$x + A := \{x + a \,|\, a \in A\}$

$A + B := \{a + b \,|\, a \in A \text{ and } b \in B\}$

# Chapter 1

# Introduction

The object of control theory is, given a plant or process with a means of input, build a control that brings about a prespecified objective. Such objectives could include rendering the state origin attractive or forcing the output of the system to track a given reference signal. In much theory on this subject, the plant is assumed known and a control is designed using this plant information. By contrast, in this thesis exact information of the plant is not assumed. Instead the plant is assumed only to belong to a class of systems, on which only structural information is available. The aim is to construct a control, using only this limited information, which will achieve the desired control objectives.

Adaptive control of uncertain systems has been studied for the past forty years, see for example the bibliography of Åström [2], and that of Ilchmann [30] for more recent work. In early work, controls would be designed to estimate some of the parameters involved in the plant model and then adjust themselves accordingly. Other methods used probing signals to see how the plant would react, and then design a control on the basis of this information.

As mentioned above, in this thesis the controls will be designed using knowledge of the structural properties of the class of system. By this we mean that bounds on uncertainties, modulo scaling, are available and/or properties of the system, such as minimum phase and relative degree, are also known. This condition is very useful in practice, as it gives rise to 'robust dynamical systems'. Such systems, but with added perturbations, will show the same type of stability properties as the original.

Since the controls are constructed using structural information of a system, a suitable control will achieve the desired control objectives for any system that possesses these structural properties, ie. the control is universal for a class of systems.

Prior to 1983, typical structural information on, for example, single-input, single-output systems of the form

$$\begin{aligned}\dot{x}(t) &= ax(t) + bu(t), \qquad x(0) \in \mathbb{R} \\ y(t) &= cx(t) \\ (a, b, c) &\in \mathbb{R} \times \mathbb{R} \times \mathbb{R}\end{aligned}$$

might be

- the relative degree of the plant is 1, ie. $cb \neq 0$.
- the sign of $cb$ is known

Clearly the first condition is reasonable, since if it were not true, either the input would have no effect on the system or no output would be available. However the second condition is restrictive. A weaker assumption would be for $cb$ to be non-zero. A.S. Morse worked on a similar problem where the sign of a certain parameter was found to be critical. He questioned in [56], whether the knowledge of the sign of the parameter was absolutely necessary? R.D. Nussbaum proved in his paper [64], that the knowledge of the sign of this parameter was not necessary. He overcame this lack of knowledge of the sign of $cb$ using an analytic function $\mathcal{N} : \mathbb{R} \to \mathbb{R}$, that has since taken his name, with the property

$$\limsup_{k\to\infty} \frac{1}{k}\int_0^k \mathcal{N}(s)\,ds = +\infty \text{ and } \liminf_{k\to\infty} \frac{1}{k}\int_0^k \mathcal{N}(s)\,ds = -\infty. \tag{1.1}$$

An example of such a function, that which Nussbaum gave, is

$$N(k) = \cos\left(\frac{\pi}{2}k\right)\exp(k^2). \tag{1.2}$$

A demonstration of this Nussbaum function in use is given in the following Willems and Byrnes example [81]. Given the system,

$$\begin{aligned}\dot{x}(t) &= ax(t) + bu(t), \qquad x(0) \in \mathbb{R} \\ y(t) &= cx(t) \\ a,\ b,\ c &\in \mathbb{R} \text{ and } cb \neq 0\end{aligned} \tag{1.3}$$

where $a$, $b$, and $c$ are unknown, the desired control objective is to render the state origin, attractive. The control $u(t) = -ky(t)$ would be viable as long as $k$ were 'large enough' ($|k| > a/|cb|$) and $k$ had the same sign as $cb$. Since the sign of this critical parameter is not known, the actual form of the control must be modified. One way of overcoming the lack of knowledge of the magnitude of $k$ is to make it a time dependent function that increases with respect to time. Ultimately, after a time $T$ say, it would exceed, and would remain larger than this critical magnitude. However, we still do not know what sign $k$ should take. This is where the Nussbaum function plays a part.

The intuition above leads to the following *adaptive* control,

$$\begin{aligned} u(t) &= \mathcal{N}(k(t))y(t) \\ \dot{k}(t) &= y^2(t), \qquad k(0) = k_0 \in \mathbb{R} \end{aligned}$$

where $\mathcal{N}(\cdot)$ is a Nussbaum function. The control makes the origin globally attractive by the following argument:

With the given control, the closed loop system is

$$\dot{x}(t) = [a + bc\mathcal{N}(k(t))]\, x(t), \qquad x(0) = x^0.$$

with solution $x(\cdot)$ defined on a maximal interval of existence $[0, \omega)$. Consider

$$y(t)\dot{y}(t) = [a + bc\mathcal{N}(k(t))]\, \dot{k}(t),$$

integration gives

$$\frac{1}{2}y^2(t) = \frac{1}{2}y^2(0) + a(k(t) - k(0)) + bc \int_{k(0)}^{k(t)} \mathcal{N}(k)\, dk.$$

Assuming, for a contradiction, $k$ is unbounded, then there is a time, $T$, such that $k(t) \geq 1$ for all $t \geq T$. From the equation immediately above, by dividing by $k(t) \geq 1$ and taking the limit infimum as $t \to \omega$ of the right hand side of the resulting inequality,

$$0 \leq \text{ constant } + \liminf_{k \to \infty} \frac{cb}{k} \int_{k(0)}^{k} \mathcal{N}(s)\, ds.$$

A contradiction is achieved by the properties of Nussbaum functions. (Recall, taking a negative-valued $cb$ through the *liminf* changes the latter to a *limsup*.) So $k(\cdot)$ is bounded. This assures the boundedness of $y$ and $x$ (which in turn gives the maximal interval of existence of the solutions as $[0, \infty)$). From $\dot{k}(t) = y^2(t)$ we get $y \in L^2$. By the boundedness of $x(\cdot)$ and $k(\cdot)$ we get $\dot{y} \in L^\infty$. These latter two properties of $y$ gives $y(t) \to 0$ as $t \to \infty$, using Theorem $A.3.3$. The fact that $x(t) \to 0$ as $t \to \infty$ follows immediately.

Many stability proofs in this thesis follow this general argument.

Helmke and Prätzel-Wolters have extensively studied the class (1.3), see [26] and [27]. Improvements to this control have been made in [28] and [14].

This control given here however, is not robust. Minor perturbations to the given system can result in the control objectives not being met. Minor perturbations bounded by known continuous functions of the output can be overcome with simple alterations in the control and adaption dynamics. For example suppose the state dynamics are of the form

$$\dot{x}(t) = ax(t) + g(t, x(t)) + bu(t), \qquad x(0) \in \mathbb{R}$$

where $|g(t,x)| \leq \rho(cx)$ for all $(t,x)$, and $\rho(\cdot)$ is a known continuous function. The control could now take on the form

$$\begin{aligned} u(t) &= \mathcal{N}(k(t))[|y(t)| + \rho(y(t))]\mathrm{sgn}(y(t)) \\ \dot{k}(t) &= [|y(t)| + \rho(y(t))]|y(t)|, \qquad k(0) \in \mathbb{R}_+ \end{aligned}$$

where

$$\mathrm{sgn}(y) = \begin{cases} +1, & y > 0 \\ 0, & y = 0 \\ -1, & y < 0 \end{cases}$$

Notice now, that the control is discontinuous at the point $y = 0$. Standard ordinary differential equations cannot cope with this analytical difficulty. A method of overcoming this is to embed the problem in a differential inclusion: $\mathrm{sgn}(y)$ is now replaced with the set-valued map $\psi(\cdot)$ defined as

$$\psi(y) = \begin{cases} \{+1\}, & y > 0 \\ [-1,+1], & y = 0 \\ \{-1\}, & y < 0. \end{cases}$$

Taking care to formulate the problem correctly, solutions can be guaranteed using Theorem $A$.3.1. Using this control, all solutions (they need not be unique) will be attracted to the origin in a similar manner as before. The idea of embedding a discontinuous ordinary differential equation in a differential inclusion will appear throughout this thesis. This is mainly due to the feedback control being discontinuous, as in the above example. The concept is also used to model some uncertainties in some of the systems studied in a similar manner to that found in [68], [70] and [73], for example.

The Willems and Byrnes example demonstrates some of the basic ideas of adaptive control of uncertain systems. However, the concepts become a little more involved as one moves to multi-dimensional state, output and input vectors. For example, consider the $m$-input, $m$-output, $n$-dimensional system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \qquad x(0) = x^0 \in \mathbb{R}^n \\ y(t) &= Cx(t) \end{aligned}$$

where $(A,B,C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n}$ and $n$ are all unknown. How can the ideas established in the Willems and Byrnes example [81] cross over to this case?

In this multi-dimensional case the property of *minimum phase* plays a central role. This is the requirement that

$$\det \begin{bmatrix} sI - A & B \\ C & 0 \end{bmatrix} \neq 0 \text{ for all } s \in \overline{\mathbb{C}}_+.$$

In the 1-dimensional case the sign of $cb$ was important. In the multi-dimensional case the eigenvalues of the matrix $CB$ are crucial. It would be preferable to have them in either the left half or right half complex plane. For example if $\sigma(CB) \subset \mathbb{C}_+$, then a control, similar to that in the one-dimensional Willems and Byrnes case above, can be used, namely

$$\begin{aligned} u(t) &= -k(t)y(t), \\ \dot{k}(t) &= \|y(t)\|^p, \qquad k(0) \in \mathbb{R},\ p \geq 1. \end{aligned}$$

However, if the structural conditions imposed on the uncertain system were to include $\sigma(CB) \subset \mathbb{C}_+$ or $\sigma(CB) \subset \mathbb{C}_-$, as in [31], then the use of Nussbaum functions would again be needed. In fact in this case a new type of Nussbaum function would be needed, called a *scaling invariant Nussbaum function*, introduced by Logemann and Owens [46]. Such a function can be defined as an analytical function $\mathcal{N}(\cdot)$ that has the property, for every $\alpha$, $\beta > 0$ the function $\overline{\mathcal{N}}(\cdot)$, defined as

$$\overline{\mathcal{N}}(t) := \begin{cases} \alpha\mathcal{N}(t), & \mathcal{N}(t) \geq 0 \\ \beta\mathcal{N}(t), & \mathcal{N}(t) < 0 \end{cases} \tag{1.4}$$

is a Nussbaum function. An example of such a function is (1.2). In this case it is needed to overcome analytical difficulties involved with the maximum and minimum of the eigenvalues of a positive definite matrix used in a Lyapunov function for the system. (Scaling invariant Nussbaum functions will be used in this thesis, mainly in the more general nonlinear systems of Chapters 4, 5 and 7.)

The structural conditions on the multi-dimensional system can be relaxed even further. The restrictive condition, $\sigma(CB) \subset \mathbb{C}_+$ or $\sigma(CB) \subset \mathbb{C}_-$, can be replaced by the property $\det(CB) \neq 0$. This ties in, more intuitively, with the original concept of $cb \neq 0$ in the one-dimensional case. From the example above, a possible strategy would be to incorporate a matrix, $M$ say, in the control ($u(t) = -k(t)My(t)$) such that $M$ *unmixes* the matrix $CB$, ie. $CBM$ has its eigenvalues in the right half complex plane. Such a matrix $M$ shall be referred to as an *unmixing matrix for the matrix* $CB$. However, $CB$ will be unknown. A finite set of matrices is therefore sought such that, given an invertible matrix, one of the members of the finite set is an unmixing matrix for the given invertible matrix. The control could then involve a cycling mechanism that continually cycles through this finite number of unmixing matrices until the correct one is located. Mårtensson has proved the following lemma.

**Lemma:** For each $m \in \mathbb{N}$, there exists a finite set of matrices $\mathcal{M}$ such that for all $U \in \mathcal{GL}(m, \mathbb{R})$, invertible, real $m \times m$ matrices, there exists an $M \in \mathcal{M}$ such that $\sigma(UM) \subset \mathbb{C}_+$.

This is clearly the required result. A control that stabilises this multi-dimensional system is

$$
\begin{aligned}
u(t) &= k(t)M(k(t))y(t), \\
\dot{k}(t) &= \|y(t)\|^p, \qquad k(0) \in \mathbb{R}
\end{aligned}
$$

for some $p \geq 1$, where $M(\cdot)$ is a suitable function that cycles through the elements of an unmixing set. This result was given by [13], [49] and completed in [33].

There are some practical considerations that should be noted. Given $m$, construction of a finite unmixing set is very difficult. Some papers have been written concerning the unmixing sets for $m = 2$ or 3, see [51] and [84]. Even in this latter case, the difficulty can be easily seen. Knowledge of unmixing sets for larger $m$ is minimal.

Often due to this difficulty, more restrictive assumptions are placed on the system. One possible replacement assumption is $\sigma(CB) \subset \mathbb{C}_-$ or $\sigma(CB) \subset \mathbb{C}_+$, allowing for the simple unmixing set $\{-I, +I\}$, where $I$ is the identity matrix. The other common possible replacement assumption is, the class of system permitted is one for which an unmixing set is known. The former is restrictive, and the latter lacks practicality. In the thesis this problem is addressed. An alternative to Mårtensson's approach is given for a class of two-dimensional systems. Also a finite unmixing set is given for general $m$-dimensional matrices on which further conditions, in addition to that of being invertible, are imposed, ie. a practical example of the second possibility is given. This latter finite unmixing set is found useful in the more general nonlinear setting met later on in the thesis where the matrices to be unmixed exhibit time-dependence.

In the analysis of linear systems with nonlinear perturbations, many of the ideas of variable structure control with sliding mode control appear (for variable structure see [85] for an introduction, [29] for a survey, as well as [73], [16], [7], [6] and [75]). In this theory the control has a variable structure in the sense that the control $u = [u_i]_{m\times 1}$ takes on different forms $u_i = u_i^+(x)$ or $u_i = u_i^-(x)$, depending on the sign of the $i^{\text{th}}$ element of a vector function $s(x) = [s_i(x)]_{m\times 1}$. The function $s(\cdot)$ is called a switching function, as the set $\{x \mid s(x) = 0\}$ (sometimes called the sliding mode) is the boundary between the areas where the different controls are active. The aim of the control is to force the state to the sliding mode in finite time. Once the state has reached this set, the control objectives are met by the sliding mode dynamics (the dynamical system restricted to the sliding mode).

For example, in the linear case with dynamics

$$
\dot{x}(t) = Ax(t) + Bu
$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $A$ a real $n \times n$ matrix and $B$ a real $n \times m$ matrix, a suitable $m \times n$ matrix $C$ is chosen with switching function

$$s(x) := [s_1(x)\ s_2(x)\ \ldots\ s_m(x)]^T = Cx = [c_1^T\ c_2^T\ \ldots\ c_m^T]^T x$$

ie. with $s_i(x) = c_i^T x$. Each $s_i(x)$ describes a hyperplane $s_i(x) = 0$ which is a switching surface. On the two sides of this surface the control takes on two different forms as mentioned above. There is usually a discontinuity in the control at this surface, a discontinuity which is clearly visible in the controls throughout this thesis. For example, in the nonlinearly perturbed linear systems of Chapters 2, 3 and 6, where the full state is available for feedback purposes, a quasi-output, $y$, is constructed and a co-ordinate transformation is made. A suitably chosen adaptive control, with a discontinuity at the set where the constructed quasi-output is zero, forces $y$ to tend to zero. With the control objective of stability, the zero dynamics of the system (the system dynamics restricted to $y = 0$) are constructed in such a way so it is asymptotically stable.

The concept of variable structure has also been used in the general nonlinear systems, see for example [22]. The form of the nonlinear systems of Chapters 4, 5 and 7 of this thesis are already in a normal form where the ideas of variable structure in the controls can be used.

However these controls with discontinuities at the surfaces $s_i(x) = 0$ lead to the phenomenon of chattering. This is the rapid switching between the various controls when the state is on the surfaces $s_i(x) = 0$, see [29], [16], [7], [6] and [85]. In the steady state, chattering appears as a high frequency oscillation about the desired equilibrium point. Efforts have been made to reduce chattering, see for example the references given above.

The ideas used in the design of the feedbacks in variable structure are similar to those used in this thesis. For example, with the control objective of stability, the intuition behind the feedbacks to be constructed is to force an output to zero, allowing the remaining state components then to decay to zero by some internal stability properties of the system (analogous to stable sliding modes).

The systems studied so far in this introduction are generally linear in nature. As discussed above the question arises as to the robustness of the control, ie. will the control still achieve the control objectives under perturbations to the system. This often depends on the availability of the state for feedback purposes. There are three main possibilities to consider. Firstly the case where the full state is available. The second and third cases are variations of the following system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B[u(t) + g(t, x(t))] + f(t, x(t)) \qquad x(0) \in \mathbb{R}^n \\ y(t) &= Cx(t) \end{aligned}$$

where $(A, B, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times p} \times \mathbb{R}^{m\times n}$, $u(t) \in \mathbb{R}^p$, $g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^p$ and $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$. The functions $g$ and $f$ are possibly nonlinear perturbations. The two distinct cases are $p = m$

and $p > m$. When $p = m$, the minimum phase condition is imposed on $(A, B, C)$. However, when $p > m$ a change of variable is made. There should exist a known $F \in \mathbb{R}^{m \times p}$ such that the system

$$\dot{x}(t) = Ax(t) + B[u(t) + g(t, x(t))] + f(t, x(t)) \qquad x(0) \in \mathbb{R}^n$$
$$\hat{y}(t) = FCx(t) = Fy(t)$$

is minimum phase, ie. $(A, B, FC)$ is minimum phase. The cases where the full state is available and the case where $p = m$ are the extremes.

Linear systems with nonlinear perturbations studied in Chapters 2 and 3 will have the full state available for feedback purposes. This will allow a wide class of functions $f(\cdot,\cdot)$ and $g(\cdot,\cdot)$ bounded in terms of a known function of the state. Thus the perturbations are more general in nature compared with the case where the full state is not available. Modifications to the theory to include the output constrained cases mentioned above will be made. In these cases the class of functions, $f(\cdot,\cdot)$ and $g(\cdot,\cdot)$, permitted will be bounded by a known function of the output (in the case where $p = m$) or the constructed output $\hat{y}$ (in the case where $p > m$).

From these foundations in adaptive control of uncertain systems, many different directions can be taken. For example, no claims so far have been made as to the optimality of the control, or the rate at which the desired control objectives are met. Work on the latter problem has been carried out by, amongst others, Ilchmann, Owens, and Logemann, see [35], [36], [34] and [45], who have guaranteed exponential decay of the state, or in the tracking case the error between the output and a given reference signal, to the control objective.

Work has also been done into the area where the relative degree of the given system is not 1. Early work by Morse [57] gave results for systems where the relative degree was either 1 or 2 and known. Later work by Morse [60] revealed that the latter knowledge was not needed for these special two cases. Mårtensson [48], Mudgett and Morse [61], and Narendra [62] have given results for general relative degree $n^*$, but here the relative degree must once more be known.

Other areas of research have been into systems without the minimum-phase condition. Mårtensson, [48] and [49], and Miller and Davison [55], [53] and [54] are notable contributors to this area. The lack of this condition means the system is not stabilisable by high gain. Instead, other conditions are imposed that render the system stabilisable for some values of $k$. More generally, in [48] Mårtensson proves that: if, for an unknown linear system $\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t)$, it is known that a (nonadaptive) regulator of order $l > 0$ stabilises the system, then it is possible to construct an adaptive stabilising controller for this unknown system.

The desired control objective might not always be to stabilise the state origin. Often the output (or the constructed output $\hat{y}$) might be required to 'track' a given reference signal. Frequently, reference signals are taken to be functions that can be generated by means of a differential equation. In this thesis, we allow very general reference signals, namely the class of functions,

$W^{1,\infty}(\mathbb{R}, \mathbb{R}^p)$. This is the class of bounded functions that are absolutely continuous on compact intervals of $\mathbb{R}$ and have essentially bounded derivatives. As well as asymptotic tracking, we shall consider another type of tracking called $\lambda$-tracking. This is where the error between the output and the reference signal is asymptotic to a prespecified ball of radius $\lambda$. Intuitively one would expect the control to switch off when the error is inside such a ball, and in practice this is how it works. Major benefit of this type of tracking are (a) noise corrupted outputs can be handled and (b) the control can be made continuous.

The main theme of this thesis is the adaptive stabilisation of uncertain *nonlinear* systems. As mentioned above, theory in this area has already been established by such authors as Ryan, Ilchmann, Owens, Logemann and Townley. The systems generally under consideration in these works are linear systems with nonlinear perturbations. These will be considered in Chapters 2 and 3 of this thesis, overcoming some of the impracticalities that can arise with the direct use of Mårtensson's lemma. These types of systems will again be considered later in the thesis, when the control objective of tracking is considered. Chapters 4, 5 and 7 will be concerned with more general nonlinear systems. For example, work has already been done on systems on the form

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t), \qquad x(0) \in \mathbb{R}$$

where $f(\cdot)$, $g(\cdot) \in C^\infty(\mathbb{R}, \mathbb{R})$, $f(0) = 0$ and $g(x) \neq 0$ for all $x \in \mathbb{R}$. Such system have been studied by such authors as Nikitin and Schmid [63], Khalil and Saberi [40], and Saberi and Lin [74]. The nonlinear systems in this thesis shall be of the form

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), y(t)), \qquad x(0) \in \mathbb{R}^n \\ \dot{y}(t) &= g(t, x(t), y(t)) + H(t, x(t), y(t))u(t), \qquad y(0) \in \mathbb{R}^m \end{aligned} \tag{1.5}$$

where $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $g : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ and $H : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{m\times m}$. When there is no explicit time dependence in the functions $f$, $g$ and $H$, this can be derived from a more standard form

$$\begin{aligned} \dot{z}(t) &= a(z(t)) + B(z(t))u(t), \qquad z(0) \in \mathbb{R}^{n+m} \\ y(t) &= c(z(t)) \end{aligned} \tag{1.6}$$

where $a : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$, $B : \mathbb{R}^{n+m} \to \mathbb{R}^{(n+m)\times m}$ and $c : \mathbb{R}^{n+m} \to \mathbb{R}^m$, via the transformation given in Isidori [39], and see Isidori and Byrnes' papers [8], [9], [11], [10] and [12]. For this transformation to exist it is sufficient that, the column vectors of the matrix $B$ ($B_i$ say, where they are considered as functions $z \mapsto B_i(z)$) be involutive and the system (1.6) has vector relative degree $\{1, 1, \dots, 1\}$ for all $z \in \mathbb{R}^{n+m}$. Some form of stability will be imposed on the zero dynamics of the system, ie. $\dot{x}(t) = f(t, x(t), 0)$. The idea, in the stabilisation case, will be to drive the output to the origin, and then let the state tend to the origin under the zero dynamics. This idea is similar to that of variable structure, as outlined earlier. Three different subclasses of the nonlinear system (1.5) will be considered, each mainly characterised by its zero dynamics. Two will have asymptotically stable zero dynamics, while the other will have exponentially stable zero dynamics. Different growth constraints will be imposed on the functions $f(\cdot,\cdot,\cdot)$, $g(\cdot,\cdot,\cdot)$ and $H(\cdot,\cdot,\cdot)$ which will be dependent on the system being studied. For example, the exponentially stable zero dynamics system will permit less restrictive bounds on these functions. Minor changes will be made to these three subclasses, when the objective of tracking is considered later in the thesis.

These systems will be gradually introduced, starting with the more elementary case where $m = 1$. The fundamental ideas involved in the theory will be introduced here, allowing the complications involved with the multi-input, multi-output systems to be dealt with separately.

The thesis will be concluded with an appendix containing established theory used during the thesis. The theory of differential inclusions, which plays a central role in overcoming the discontinuities in the controls, will be discussed at length. The Inverse Lyapunov theory, used in the chapters concerning the Isidori type nonlinear systems, is also covered. The relevant results on Uniform Distribution theory, used in Chapters 2 and 6, are briefly discussed.

# Chapter 2

# Stabilisation of nonlinearly perturbed linear systems: two-input case

## 2.1 Introduction

The problem of high-gain adaptive stabilisation of uncertain dynamical systems has been studied by many authors. This is the stabilisation of any system, within a pre-defined class of systems, with the use of an adaptive gain and an output feedback.

Consider the class $\mathcal{L}$ of finite-dimensional, single-input, single-output linear systems. As discussed in Chapter 1, it is well established that, for the subclass $\mathcal{S}$ consisting of all elements of $\mathcal{L}$ that have the properties of minimum phase and relative degree one, there exist a $\mathcal{S}$-universal stabiliser: a single adaptive linear output feedback strategy that, for every member (unknown to the controller) of the subclass $\mathcal{S}$, renders the zero state globally attractive. Many extensions of this result to the single-input nonlinear systems have been accomplished (see [30] for a survey and comprehensive bibliography).

The transition from the single-input to the multiple-input case is not straightforward. For the class $\mathcal{S}$ of $m$-input, $m$-output, finite dimensional, linear, minimum-phase, relative-degree-one (that is $\det(CB) \neq 0$) systems of the form $(C, A, B)$, the existence of spectrum unmixing sets of finite cardinality, conjectured in [13] and proved by Mårtensson [49] (see also [50]) plays a central role.

Exploiting this result, $\mathcal{S}$-universal stabilisers have been constructed, see [50] and [33]. In effect, the problem of lack of knowledge of the invertible 'high-frequency' gain $CB$ can be circumvented by a controller design which cycles through an explicit spectrum-unmixing set for $CB \in \mathcal{GL}(m;\mathbb{R})$. This chapter is in a similar vein - but focuses on *nonlinear* two-input systems and presents an approach with implicit recourse to spectrum unmixing sets.

Specifically, systems of the following general form will be studied:

$$M^* z^{(p)}(t) + B^* u(t) = g(t, z(t), \dot{z}(t), \ldots, z^{(p-1)}(t)), \qquad z(t),\, u(t) \in \mathbb{R}^2 \tag{2.1}$$

where $M^*$, $B^*$ and $g$ (assumed measurable in $t$ and continuous in $(z, \dot{z}, \ddot{z}, \ldots, z^{(p-1)})$) are uncertain. The following structural assumptions specifically classify the subclass of system (2.1) that is to be studied.

**Structural Assumptions.**

**A1:** $M^*$, $B^* \in \mathcal{GL}(2;\mathbb{R})$.

**A2:** $g$ is bounded, modulo an unknown scalar multiplier $\mu > 0$, by a known continuous function $\gamma$ of the state in the sense that, for almost all $t \in \mathbb{R}$,

$$\|g(t, v_1, v_2, \ldots, v_p)\| \le \mu\gamma(v_1, v_2, \ldots, v_p) \qquad \forall (v_1, v_2, \ldots, v_p) \in \mathbb{R}^{2p}.$$

Defining the set-valued map

$$\mathcal{Z} : (v_1, v_2, \ldots, v_p) \mapsto \gamma(v_1, v_2, \ldots, v_p)\overline{\mathbb{B}}_2,$$

and writing $M = \mu^{-1}M^*$, $B = \mu^{-1}B^*$, we re-interpret (2.1) in the following generalised sense

$$M z^{(p)}(t) + B u(t) \in \mathcal{Z}(z(t), \dot{z}(t), \ddot{z}(t), \ldots, z^{(p-1)}(t)), \qquad M,\, B \in \mathcal{GL}(2;\mathbb{R}), \tag{2.2}$$

ie. (2.1) is embedded in the differential inclusion (2.2). Thus, the analytical framework is that of differential inclusions, see [3] and [18] (which is also appropriate to the essentially discontinuous nature of the ensuing feedback strategy).

A basic question may now be posed: does there exist an adaptive feedback strategy that guarantees that, for every pair of invertible matrices $M$, $B$ (unknown to the controller) and for every initial condition

$$(z(t_0), \dot{z}(t_0), \ddot{z}(t_0), \ldots, z^{(p-1)}(t_0)) = \zeta^0 \in \mathbb{R}^{2p},$$

every solution of the initial-valued problem (2.2) is bounded and tends to the zero state as $t \to \infty$?

This chapter answers this question in the affirmative by an explicit construction of one such universal stabiliser. In particular, on the introduction of an appropriate linear coordinate transformation

$$T : (z(t), \dot{z}(t), \ddot{z}(t), \ldots, z^{(p-1)}(t)) \mapsto (w(t), y(t)) \in \mathbb{R}^{2(p-1)} \times \mathbb{R}^2,$$

see [30], we show that a strategy of the following form suffices

$$\begin{aligned} u(t) &\in k^2(t)[\gamma(T^{-1}(w(t), y(t))) + \|y(t)\|] \begin{bmatrix} \cos k(t) & -\sin k(t) \\ \sin k(t) & \cos k(t) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & s(k(t)) \end{bmatrix} \psi(y(t)) \\ \dot{k}(t) &= \gamma(T^{-1}(w(t), y(t)))\|y(t)\| + \|y(t)\|^2, \qquad k(0) = k^0 \end{aligned} \tag{2.3}$$

where

$$\psi : y \mapsto \begin{cases} \{\|y\|^{-1}y\}, & y \neq 0 \\ \overline{\mathbb{B}}_2, & y = 0 \end{cases}$$

Note that $k \mapsto k^2 \cos(k)$ is a continuous Nussbaum function (in the same sense as that of [64]), see (1.1).

Choice of the function $s(\cdot)$ in (2.3) is a major component of the chapter. We discuss in detail two such choices which ensure the following:

*For every* $(\zeta^0, k^0) \in \mathbb{R}^{2p+1}$,

*(i) there exists a solution of the feedback-controlled initial-value problem (2.2-2.8);*

*(ii) every solution has maximal interval of existence* $[t_0, \infty)$ ;

*(iii)* $\lim_{t\to\infty} k(t)$ *exists and is finite;*

*(iv)* $(z(t), \dot{z}(t), \ddot{z}(t), \ldots, z^{(p-1)}(t)) \to (0, 0, 0, \ldots, 0)$ *as* $t \to \infty$.

## 2.2 Universal adaptive stabiliser

### 2.2.1 Co-ordinate transformation

In the given system (2.2), the full state is available for feedback purposes. A co-ordinate transform is given to transform the system into a normal form with a constructed output $y(\cdot)$ and a sub-state vector $w(\cdot)$. The latter is constructed so the control does not explicitly appear in its dynamics. Also the zero dynamics of the dynamical equation of the sub-state are made to be stable by a suitable choice of output. The control enters the system through the dynamics of the constructed output, ie. a cascade system is made. The idea will be to drive the constructed output to zero under the effects of the control, and have the internal dynamics of this constructed state drive the rest of the system to the origin. This is the approach used in [70], [73], [68] and [33]. A co-ordinate transformation which will transform the system (2.2) to a normal form with these properties can be constructed in the following way.

Choose matrices $C_i \in \mathbb{R}^{2\times 2}$, $i = 1, 2, \cdots, p-1$ such that all the eigenvalues of the linear system

$$S: \qquad z^{(p-1)}(t) + C_{p-1}z^{(p-2)}(t) + \cdots + C_2\dot{z}(t) + C_1 z(t) = 0$$

lie in the open left half complex plane $\mathbb{C}_-$. Define a co-ordinate transformation, $T$, by

$$T : (z(t), \dot{z}(t), \cdots, z^{(p-1)}(t)) \mapsto (w(t), y(t))$$

where

$$w(t) = (w_1(t), w_2(t), \cdots, w_{p-1}(t)) = (z(t), \dot{z}(t), \cdots, z^{(p-2)}(t)) \in \mathbb{R}^{2(p-1)} \tag{2.4}$$

and

$$y(t) = C_1 z(t) + C_2 \dot{z}(t) + \cdots + C_{p-1} z^{(p-2)}(t) + z^{(p-1)}(t) \in \mathbb{R}^2. \tag{2.5}$$

This transformation takes (2.2) into the form

$$\left.\begin{aligned} &\dot{w}(t) = L_1 w(t) + L_2 y(t) \\ &M[\dot{y}(t) + L_3 w(t) - C_{p-1} y(t)] + Bu(t) \in \mathcal{F}(w(t), y(t)) \\ &(w(0), y(0)) = T\zeta^0 \end{aligned}\right\} \tag{2.6}$$

where $\mathcal{F} := \mathcal{Z} \circ T^{-1}$ and $L_1, L_2, L_3$ are linear with

$$L_1 : (w_1, w_2, \ldots\ldots, w_{p-2}, w_{p-1}) \mapsto (w_2, w_3, \ldots., w_{p-1}, -\sum_{i=1}^{p-1} C_i w_i)$$

The spectrum of $L_1$ is precisely that of the linear system $S$ and so $\sigma(L_1) \subset \mathbb{C}_-$. Therefore the zero dynamics of (2.6), ie. $\dot{w}(t) = L_1 w(t)$ are stable. Also, as required, the control only explicitly appears in the dynamics of the constructed output.

### 2.2.2 Adaptive feedback strategy for the special case $|M^{-1}B| > 0$

As seen in the examples of Chapter 1, when generality is sought in the class of system to be stabilised, the control can quickly take on a complicated form. Therefore the chapter will proceed in two stages, beginning with the special case $|M^{-1}B| > 0$. A control strategy is constructed that stabilises the system (2.6), but with the assumption $\det(M^{-1}B) \neq 0$ replaced with this stronger condition. This will develop the basic ideas behind the adaptive control, without being involved with the complexities that result from the more general case.

As found in other works in this area, notably [70], [73] and [33], a discontinuous control is adopted. However, in an analytical framework this presents a problem as ordinary differential equation theory cannot cope with discontinuous systems. Differential inclusions provide a possible solution to these analytical difficulties. The idea is to embed the feedback system in a differential inclusion, with appropriate properties that guarantee the existence of a solution. However a solution of the inclusion is not necessarily unique, so any result established by the analysis must hold for all the possible solutions. Therefore, initially the control is presented in a discontinuous form, but is later embedded into a set inclusion for these analytical reasons.

The control is a special case of that presented by Ryan in [70] and [73]. Comparisons can quickly be made due to the similarity of the class of system under consideration. Define on $\mathbb{R}$ the map $\mathcal{O}_+(\cdot)$

$$\mathcal{O}_+(k) := \begin{bmatrix} \cos(k) & -\sin(k) \\ \sin(k) & \cos(k) \end{bmatrix}.$$

This is the two-dimensional rotation matrix of $k$ radians about the origin. The properties, see [51], of these matrices shall play an important role in the development of the controls. In the control below, they replace the more conventional cycling matrix that cycles through a known finite set of unmixing matrices. This time-varying rotation matrix appears in the control with the argument $k$ (driven by the state), dictating the rate at which the rotation matrices are traversed.

With these ideas, the control can formally be given as

$$u(t) = k^2(t)[f(w(t), y(t)) + \|y(t)\|]\mathcal{O}_+(k(t))y(t)/\|y(t)\|$$

and

$$\dot{k}(t) = f(w(t), y(t))\|y(t)\| + \|y(t)\|^2, \qquad k(t_0) = k^0 \in \mathbb{R}$$

where $f : (w, y) \mapsto \max\{\|\phi\| \mid \phi \in \mathcal{F}(w, y)\}$. The latter function is required in the control to compensate for the nonlinear perturbations that appear in the given system. The control has a discontinuity at the point $y = 0$, so the analytical difficulties that could arise from this can be overcome by embedding the control into a set inclusion, see Filippov [18] for a detailed guide in constructing suitable set-valued maps. Here a simple construction is used, which embeds the discontinuous function $y \mapsto y/\|y\|$ into an upper-semicontinuous set-valued map with convex and compact values. The map is namely $y \mapsto \psi(y) \subset \mathbb{R}^2$ defined as

$$\psi(y) := \begin{cases} \{\|y\|^{-1}y\}, & y \neq 0 \\ \overline{\mathbb{B}}, & y = 0 \end{cases}. \tag{2.7}$$

Clearly this has the correct properties, see Theorem $A.2.3$. These properties are needed to guarantee solutions to the feedback system. Therefore, letting $x = (w, y, k)$, the adaptive control strategy is now given as

$$\left.\begin{aligned} u(t) &\in \Psi(x(t)) := k^2(t)[f(w(t), y(t)) + \|y(t)\|]\mathcal{O}_+(k(t))\psi(y(t)) \\ \dot{k}(t) &= f(w(t), y(t))\|y(t)\| + \|y(t)\|^2, \qquad k(t_0) = k^0 \in \mathbb{R} \end{aligned}\right\} \tag{2.8}$$

where $f(\cdot,\cdot)$ is defined as above. The continuity of the set-valued map $\mathcal{F}$, together with compactness of its values ensures that $f(\cdot,\cdot)$ is a well-defined continuous map $\mathbb{R}^{2p} \to [0,\infty)$ by Lemma $A.2.7$ in the appendix. This in turn, using the Lemma $A.2.6$ in the appendix, ensures the upper-semicontinuity of the set-valued map $\Psi(\cdot)$. $\Psi(\cdot)$ clearly has convex and compact values.

The overall adaptively controlled system may now be embedded in the following initial-value problem in $\mathbb{R}^N$; $N := 2p + 1$.

$$\dot{x}(t) \in F(x(t)), \qquad x(t_0) = x^0 = (T\zeta^0, k^0) \tag{2.9}$$

with the set-valued map $x \mapsto F(x) \subset \mathbb{R}^N$ defined by

$$F(x) := F_1(x) \times F_2(x) \times F_3(x)$$

where

$$\begin{aligned} F_1(x) &:= \{L_1 w + L_2 y\} \\ F_2(x) &:= \{M^{-1}[\phi - Bu] - L_3 w + C_{p-1} y \mid \phi \in \mathcal{F}(w,y), u \in \Psi(x)\} \\ F_3(x) &:= \{\|y\|^2 + f(w,y)\|y\|\} \end{aligned}$$

for all $x = (w, y, k) \in \mathbb{R}^{2(p-1)} \times \mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^N$.

To guarantee solutions of this differential inclusion, certain conditions are sufficient, see [3, Chapter 2, Theorem 3]. The set-valued map $F(\cdot)$ is upper-semicontinuous, the values of $F(\cdot)$ are non-empty, compact and convex. Clearly, for every $x$ the set $F(x)$ is non-empty. The sets $F_1(x)$ and $F_2(x)$ are both convex and compact, since they are sets with but one element. For every $x$, the sets $\Psi(x)$ and $\mathcal{F}(x)$ are non-empty, compact and convex by the way in which they were constructed. Hence for each $x$, $F(x)$ is non-empty, compact and convex. All that remains to show is the upper-semicontinuity of the set-valued map $F(\cdot)$. The upper-semicontinuity of the maps $F_1(\cdot)$ and $F_3(\cdot)$ is trivial. The functions $\mathcal{F}(\cdot)$ and $\Psi(\cdot)$ are upper-semicontinuous maps by Lemma $A.2.7$ and Lemma $A.2.6$ respectively, and this in turn gives the upper-semicontinuity of the map $F_2(\cdot)$ by Lemma $A.2.5$. The result Lemma $A.2.2$ in the appendix gives the upper-semicontinuity of the map $F(\cdot)$.

From [3, Theorem 2.1.3] and [18], see Theorem $A.3.1$, for each $x^0 \in \mathbb{R}^N$ equation (2.9) admits a solution, and every solution has a maximal extension, (see Theorem $A.3.2$ in the appendix).

### Stability analysis

For the class of systems (2.6), with the restriction $|M^{-1}B| > 0$, the following shall be proved. With arbitrary $k^0$ and $\zeta^0$ (i) every solution of (2.6) can be extended indefinitely, (i.e. finite escape times do not occur), (ii) the adaptive gain $k(t)$ tends to a finite limit, and (iii) every solution of (2.6) tends to the zero state.

**Theorem 2.2.1** *Let $x(\cdot) = (w(\cdot), y(\cdot), k(\cdot)) : [t_0, \omega) \to \mathbb{R}^N$ be a maximal solution of (2.9). Then*
*(i)* $\omega = \infty$
*(ii)* $\lim_{t\to\infty} k(t)$ *exists, and is finite*
*(iii)* $\lim_{t\to\infty} \|(w(t), y(t))\| = 0$

**Proof**

Let $D = M^{-1}B$. Since $D$ is invertible, polar decomposition gives the existence of a unique, symmetric, positive definite matrix $Q$ and an orthogonal matrix $O$ such that

$$D = QO$$

where $O$ is an orthogonal matrix with $|O| = +1$ (since $D$ has a positive determinant), and therefore has the form $O = \mathcal{O}_+(\beta)$ for some unknown $\beta \in [0, 2\pi)$. See [76].

The matrix $Q$ is used in the construction of a Liapunov function for the system. Also, since $\sigma(L_1) \subset \mathbb{C}_-$, Liapunov theory gives the existence of a unique, symmetric positive-definite matrix $P$ such that

$$PL_1 + L_1{}^T P = -I$$

This is also used in the construction of a Liapunov function for the system in the following way. Let

$$W_1 : w \mapsto \frac{1}{2}\langle w, Pw\rangle \qquad \text{and} \qquad W_2 : y \mapsto \frac{1}{2}\langle y, Q^{-1}y\rangle$$

and

$$W : x = (w, y, k) \mapsto W_1(w) + W_2(y).$$

In the analysis multiples of $w$ and $y$ are encountered. (The following result can be found in [69], and is repeated in Lemma $D.1.1$ in the appendix of this thesis.) Since $\dot{w}(t) = L_1 w(t) + L_2 y(t)$ and $\sigma(L_1) \subset \mathbb{C}_-$, there exists constants $c_0$ and $c_1$ such that for all $t_0, t \in [0, \omega)$ with $t \geq t_0$

$$\int_{t_0}^{t} \|y(s)\|\|w(s)\|\, ds \leq c_0\|w(t_0)\|^2 + c_1 \int_{t_0}^{t} \|y(s)\|^2\, ds \leq c_0\|w(t_0)\|^2 + c_1(k(t) - k(t_0)) \tag{2.10}$$

The latter inequality uses $\|y(t)\|^2 \leq f(w(t), y(t))\|y(t)\| + \|y(t)\|^2 = \dot{k}(t)$.

Write $c_2 = \|Q^{-1}C_{p-1}\| + \|Q^{-1}M^{-1}\|$. So for all $x = (w, y, k) \in \mathbb{R}^N$ with $y \neq 0$, we have, for all $\eta \in F_2(x)$,

$$\begin{aligned}
\langle \nabla W_2(y), \eta\rangle &\leq -\langle Q^{-1}y, L_3 w\rangle + c_2[f(w, y)\|y\| + \|y\|^2] \\
&\qquad - k^2[f(w, y) + \|y\|]\|y\|^{-1} \langle y, Q^{-1}D\mathcal{O}_+(k)y\rangle \\
&= -\langle Q^{-1}y, L_3 w\rangle + [f(w, y)\|y\| + \|y\|^2](c_2 - k^2\cos(k + \beta)),
\end{aligned}$$

since $\langle y, O_+(k+\beta)y\rangle = \cos(k+\beta)\|y\|^2$. (Clearly for all $\eta \in F_2(x)$ with $y=0$, $\langle \nabla W_2(y), \eta\rangle = 0$). Therefore for almost all $t$,

$$\frac{dW_2(y(t))}{dt} \le \|Q^{-1}L_3\| \int_{t_0}^{t} \|y(t)\|\|w(t)\| + \int_{t_0}^{t} (c_2 - k^2(t)\cos(k(t)+\beta))\dot{k}(t)\,dt.$$

Integrating this over the interval $[t_0, t)$, and using the adaptive dynamics,

$$\dot{k}(t) = f(w(t), y(t))\|y(t)\| + \|y(t)\|^2,$$

it follows that

$$\begin{aligned}
0 \le W_2(y(t)) &\le W_2(y(t_0)) + \|Q^{-1}L_3\| \int_{t_0}^{t} \|y(s)\|\|w(s)\|\,ds + c_2(k(t)-k(t_0)) \\
&\quad - \int_{t_0}^{t} k^2(s)\cos(k(s)+\beta)\dot{k}(s)\,ds \\
&\le W_2(y(t_0)) + c_0\|Q^{-1}L_3\|\|w(t_0)\|^2 + (c_1\|Q^{-1}L_3\| + c_2)(k(t)-k(t_0)) \\
&\quad - \int_{t_0}^{t} k^2(s)\cos(k(s)+\beta)\dot{k}(s)\,ds. \qquad (2.11)
\end{aligned}$$

Such expressions are found in the analysis of simpler plants in the area of adaptive control, see for example [70], [73] and [81]. We proceed with a similar argument to show the boundedness of the adaptive gain $k(\cdot)$. Assume for a contradiction that the monotone increasing function $k(\cdot)$ is unbounded. Let $T \in [t_0, \omega)$ be such that $k(t) \ge 1$ for $t \ge T$. Then for such $t$

$$0 \le \liminf_{t\uparrow\omega} \frac{W_2(y(t))}{k(t)} \le c^* - \limsup_{t\uparrow\omega} \frac{1}{k(t)} \int_{t_0}^{t} k^2(s)\cos(k(s)+\beta)\dot{k}(s)\,ds \qquad (2.12)$$

where $c^* := W_2(y(t_0)) + c_0\|Q^{-1}L_3\|\|w(t_0)\|^2 + (c_1\|Q^{-1}L_3\| + c_2)$. By a change of variable we get

$$\limsup_{t\uparrow\omega} \frac{1}{k(t)} \int_{t_0}^{t} k^2(s)\cos(k(s)+\beta)\dot{k}(s)\,ds = \limsup_{\theta\uparrow\infty} \frac{1}{\theta} \int_{k(t_0)}^{\theta} r^2\cos(r+\beta)\,dr = \infty$$

(The latter equality is true since $\theta^2\cos(\theta+\beta)$ is a Nussbaum function for any $\beta$, see [64], [70], [73], [58], [59] and [28] for Nussbaum functions). This result and (2.12) gives a contradiction. Hence $k(\cdot)$ is bounded.

To show that a solution of the differential inclusion (2.9) exists on the semi-infinite interval $[t_0, \infty)$, it suffices to prove the boundedness of $w(\cdot)$ and $y(\cdot)$ (Theorem $A$.3.3 in the Appendix). Define

$$k^* := \frac{1}{4} + \|Q^{-1}M^{-1}\| + \|Q^{-1}C_{p-1}\| + (\|PL_2\| + \|Q^{-1}L_3\|)^2.$$

For all $\phi \in F(x)$

$$\begin{aligned}
\langle \nabla W(x), \phi \rangle &\leq \langle Pw, L_1 w\rangle + \langle Pw, L_2 y\rangle - \langle Q^{-1}y, L_3 w\rangle + \langle Q^{-1}y, C_{p-1}y\rangle \\
&\quad + f(w,y)\|Q^{-1}M^{-1}\|\|y\| - k^2\cos(k+\beta)[f(w,y)\|y\| + \|y\|^2] \\
&\leq -\frac{1}{2}\|w\|^2 + (\|PL_2\| + \|Q^{-1}L_3\|)\|y\|\|w\| + \|Q^{-1}C_{p-1}\|\|y\|^2 \\
&\quad + f(w,y)\|Q^{-1}M^{-1}\|\|y\| - k^2\cos(k+\beta)[f(w,y)\|y\| + \|y\|^2] \\
&\leq -\frac{1}{4}(\|w\|^2 + \|y\|^2) + (k^* - k^2\cos(k+\beta))[f(w,y)\|y\| + \|y\|^2]
\end{aligned}$$

since $\langle w, PL_1 w\rangle = \frac{1}{2}\langle w, (PL_1 + L_1^T P)w\rangle$, and using (2.11). So

$$\frac{d}{dt}(W(x(t))) \leq (k^* - k^2(t)\cos(k(t)+\beta))\dot{k}(t)$$

for almost all $t$. Under integration this gives

$$W(x(t)) \leq W(x(t_0)) + \int_{k(t_0)}^{k(t)} (k^* - \theta^2\cos(\theta+\beta))\, d\theta.$$

From this and the boundedness of $k(\cdot)$, $W(x(\cdot))$ is bounded. By the definition of $W(\cdot)$, we can conclude that $w(\cdot)$ and $y(\cdot)$ are bounded. This and $k(\cdot)$ being bounded ensures $x(\cdot)$ is bounded, which gives $\omega = \infty$ by Theorem $A.3.3$ in the appendix.

All that remains to prove is $(w(t), y(t)) \to (0,0)$ as $t \to \infty$. With $x = (w, y, k)$, define the function

$$x \mapsto V(x) := W(x) - \int_{k_0}^{k} (k^* - \theta\cos(\theta+\beta))\, d\theta$$

Noting that

$$\langle \nabla V(x), \phi\rangle \leq -\frac{1}{4}(\|w\|^2 + \|y\|^2), \qquad \forall \phi \in F(x)$$

we can conclude that $(w(t), y(t)) \to (0,0)$ as $t \to \infty$, by Theorem $A.3.7$ in the Appendix. □

### 2.2.3 Adaptive feedback strategy in the general case $|M^{-1}B| \neq 0$

In the restrictive case of (2.2) with the replacement assumption $|M^{-1}B| > 0$, the basic ideas that will be used in the general case with $\det(M^{-1}B) \neq 0$ have been established. Let $\mathcal{S}$ denote the map

$$s \mapsto \mathcal{S}(s) := \begin{bmatrix} 1 & 0 \\ 0 & s \end{bmatrix}.$$

We have shown the adaptive control in the special case $|M^{-1}B| > 0$ had the form

$$\begin{aligned}
u(t) &\in k^2(t)[f(w(t), y(t)) + \|y(t)\|]\mathcal{O}_+(k)\mathcal{S}(1)\psi(y(t)) \\
\dot{k}(t) &= f(w(t), y(t))\|y(t)\| + \|y(t)\|^2
\end{aligned}$$

It can be shown, with an entirely analogous argument, that a similar control structure, but with $\mathcal{S}(-1)$ replacing $\mathcal{S}(1)$, will work in the other special case where $|M^{-1}B| < 0$. In the general case when $|M^{-1}B| \neq 0$, a continuous method of alternating between the two special cases is sought: precisely, we seek a continuous function $k \mapsto s(k) \in [-1, 1]$ such that, on replacing $\mathcal{S}(1)$ by $\mathcal{S}(s(k(t)))$, we arrive at a universal stabiliser for the general case of invertible $M^{-1}B$.

It will be proved that any continuous function $s(\cdot)$ of the following type has the required effect of stabilising the system. Let $(\tau_n)_{n\in\mathbb{N}}$ be a sequence with the following properties:

**(i)** $\tau_1 > 1$,

**(ii)** For some fixed $\rho > 3/2$ (a control parameter), $\tau_{n+1} > \tau_n^\rho$.

**(iii)** For every $\beta \in [0, 2\pi)$ there exists $\epsilon^* \in (0, \pi/2)$ and subsequences of $(\tau_{2n})_{n\in\mathbb{N}}$ and $(\tau_{2n-1})_{n\in\mathbb{N}}$, call them $(\tau_{2n_l})_{l\in\mathbb{N}}$ and $(\tau_{2n_m-1})_{m\in\mathbb{N}}$, such that

$$\{\tau_{2n_l} + \beta\}_{2\pi} \in [\epsilon^*, \pi - \epsilon^*] \quad \text{for all } l$$

and

$$\{\tau_{2n_m-1} + \beta\}_{2\pi} \in [\epsilon^*, \pi - \epsilon^*] \quad \text{for all } m.$$

(Notation: $\{a\}_b = a \bmod b$, that is $\{a\}_b$ is the unique element $c$ of $[0, b)$ such that $a = c + bd$ for some integer $d \in \mathbb{Z}$.)

The existence of such a sequence is shown constructively below.

Using the sequence $(\tau_n)_{n\in\mathbb{N}}$, the function $s(\cdot)$ may now be defined. Choose $\xi \in (0, \tau_1)$ sufficiently small so that $\tau_2 > \tau_1 + \xi$. Define the continuous function $s : \mathbb{R} \to [-1, 1]$ as follows

$$k \mapsto s(k) := \begin{cases} -1, & k \in (-\infty, \tau_1 + \xi) \cup \left(\cup_{n\in\mathbb{N}}[\tau_{2n-1} + \xi, \tau_{2n})\right), \\ +1, & k \in \cup_{n\in\mathbb{N}}[\tau_{2n} + \xi, \tau_{2n+1}) \\ \frac{1}{\xi}(2k - 2\tau_{2n} - \xi), & k \in \cup_{n\in\mathbb{N}}[\tau_{2n}, \tau_{2n} + \xi) \\ \frac{1}{\xi}(-2k + 2\tau_{2n-1} + \xi), & k \in \cup_{n\in\mathbb{N}}[\tau_{2n-1}, \tau_{2n-1} + \xi) \end{cases} \tag{2.13}$$

Figure $(2-1)$ depicts the possible evolution of the function $s(\cdot)$.

For all sufficiently large $n$, $\tau_{n+1} > (\tau_n + \xi)^{\rho^*}$ for all $\rho^* \in (3/2, \rho)$ by the following argument:

$$\tau_{n+1} - (\tau_n + \xi)^{\rho^*} > (\tau_n)^\rho - (\tau_n + \xi)^{\rho^*} \to \infty \text{ as } n \to \infty$$

since $\tau_n \to \infty$ and $\rho > \rho^*$.

Using the function $s(\cdot)$ constructed above the following adaptive strategy, based on the control

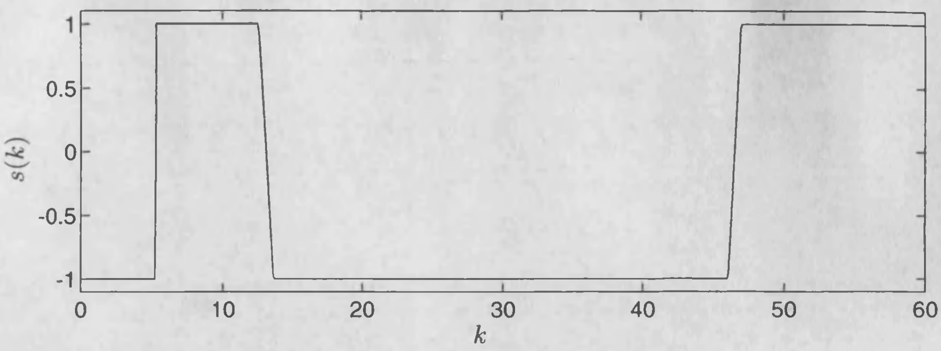


Figure 2-1: The evolution of a possible $s(\cdot)$.

strategy for the special case $|M^{-1}B| > 0$, is proposed

$$
\begin{aligned}
&u(t) \in \Psi(x(t)) := k^2(t)[f(w(t), y(t)) + \|y(t)\|]\mathcal{O}_+(k(t))\mathcal{S}(s(k(t)))\psi(y(t)) \\
&\psi(y) = \begin{cases} \{\|y\|^{-1}y\}, & y \neq 0 \\ \overline{\mathbb{B}}_2, & y = 0 \end{cases} \\
&\dot{k}(t) = f(w(t), y(t))\|y(t)\| + \|y(t)\|^2\ , \ k(0) = k^0 \in \mathbb{R}
\end{aligned}
\tag{2.14}
$$

As before $f$ is the continuous map $f : (w, y) \mapsto \max\{\|\phi\| \mid \phi \in \mathcal{F}(w, y)\}$.

The control cycles between the two strategies constructed for the two special cases. When the general control is of the correct type, it must compensate for the earlier times when the control is of the incorrect type. This is why care is taken in constructing the sequences $(\tau_n)_{n\in\mathbb{N}}$. The reader might find the references [70], [73] and [33] useful as other areas where care is needed in the construction of sequences for similar reasons. Due to the uncertainty of $\beta$, extra properties are required for the sequence in this setting.

#### Stability analysis

The stability analysis for the general case with $\det(M^{-1}B) \neq 0$ is now given. Similar ideas are used, as found in the special case analysis, and will be referenced where necessary. Care will be taken to demonstrate the differences in approach between the general and the special case.

As before, the following shall be proved. For arbitrary $\zeta^0$ and $k_0$ (i) every solution of (2.9) can be extended indefinitely (i.e. finite escape times do not occur), (ii) the adaptive gain $k(t)$ tends to a finite limit, and (iii) every solution of (2.6) tends to the zero state.

**Theorem 2.2.2** *Let $x(\cdot) = (w(\cdot), y(\cdot), k(\cdot)) : [0, \omega) \to \mathbb{R}^N$ be a maximal solution of (2.9). Then*
*(i) $\omega = \infty$*
*(ii) $\lim_{t\to\infty} k(t)$ exists, and is finite*
*(iii) $\lim_{t\to\infty} \|(w(t), y(t))\| = 0$*

**Proof:** Let $D, Q$ and $O$ be as in Theorem 2.2.1. Note however in the general case, the orthogonal matrix $O$ takes one of two forms

$$\begin{bmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{bmatrix} =: \mathcal{O}_+(\beta) \quad \text{or} \quad \begin{bmatrix} \cos(\beta) & -\sin(\beta) \\ -\sin(\beta) & -\cos(\beta) \end{bmatrix} = \mathcal{S}(-1)\mathcal{O}_+(\beta)$$

for some $\beta \in [0, 2\pi)$, unknown. The former case corresponds to $|O| = +1$, the latter to $|O| = -1$: thus, in either case, $O$ has the form $\mathcal{S}(|O|)\mathcal{O}_+(\beta)$.

Let $W_1$, $W_2$ and $W$ be as in Theorem 2.2.1. Define

$$H(k) := \begin{cases} -k^2\cos(k+\beta), & |O|s(k) = +1 \\ -k^2\cos(k+\beta) + 2k^2, & |O|s(k) \in [-1, +1) \end{cases}$$

When $|O|s(k) = 1$, $H(k)$ has the form of a Nussbaum function, but when $|O|s(k) \in [-1, 1)$, $H(k)$ does not have this form. The alternating function $s(k)$ has been constructed in such a way to mimic the Nussbaum properties of the function $H(k)$.

Using similar analysis to that found in the special case of $|M^{-1}B| > 0$, for all $y$,

$$\begin{aligned} -k^2\left\langle y, Q^{-1}D\mathcal{O}_+(k)\mathcal{S}(s(k))y\right\rangle &= -k^2\langle y, \mathcal{S}(|O|)\mathcal{O}_+(k+\beta)\mathcal{S}(s(k))y\rangle \\ &\leq H(k)\|y\|^2. \end{aligned}$$

Therefore, for all $x = (w, y, k) \in \mathbb{R}^N$ we have

$$\langle \nabla W_2(y), \eta\rangle \leq -\langle Q^{-1}y, L_3w\rangle + (c_2 + H(k))[f(w,y)\|y\| + \|y\|^2] \qquad \forall \eta \in F_2(x)$$

where the scalar $c_2$ is the same as that found in the proof of Theorem 2.2.1. Again the proof proceeds to show the boundedness of $k(\cdot)$. From the above inequality it follows, for almost all $t \in [0, \omega)$,

$$\frac{d}{dt}(W_2(y(t))) \leq -\langle Q^{-1}y(t), L_3w(t)\rangle + (c_2 + H(k(t)))\dot{k}(t)$$

which gives

$$0 \leq W_2(y(t)) \leq a_1 + a_2(k(t) - k(t_0)) + \int_{t_0}^{t} H(k(t))\dot{k}(t)\,dt \tag{2.15}$$

where $a_1 = W_2(y(t_0)) + c_0\|Q^{-1}L_3\|\|w(t_0)\|^2$ and $a_2 = c_1\|Q^{-1}L_3\| + c_2$ with $c_0$ and $c_1$ as in Theorem 2.2.1. Assume for a contradiction that the monotone increasing function $k(\cdot)$ is unbounded, and so using $\dot{k}(t) \geq 0$

$$\liminf_{t\uparrow\omega} \frac{1}{k(t)}\int_{t_0}^{t} H(k(t))\dot{k}(t)\,dt = \liminf_{\sigma\to\infty} \frac{1}{\sigma}\int_{k(t_0)}^{\sigma} H(\theta)\,d\theta.$$

Define sequences $(S_n)$ and $(T_n)$ (subsequences of the sequence $(\tau_n)$) as follows

$$S_n = \begin{cases} \tau_{2n+1}, & |O| = +1 \\ \tau_{2n}, & |O| = -1 \end{cases} \qquad T_n = \begin{cases} \tau_{2n}, & |O| = +1 \\ \tau_{2n-1}, & |O| = -1 \end{cases}.$$

Then, for $n^* \in \mathbb{N}$ large enough, consider

$$\begin{aligned}\liminf_{\sigma\to\infty} \frac{1}{\sigma}\int_{k(t_0)}^{\sigma} H(\theta)\,d\theta &\le \liminf_{N\to\infty} \frac{1}{S_{N+1}} \sum_{n=n^*}^{N} \left[\int_{S_n}^{S_{n+1}} -\theta^2\cos(\theta+\beta)d\theta + 2\int_{S_n}^{T_n+\xi} \theta^2 d\theta\right] \\ &\le \liminf_{N\to\infty} \left(-S_{N+1}\sin(S_{N+1}+\beta) + \frac{2}{3}\frac{(T_N+\xi)^3}{S_{N+1}}\right) + \text{ const.}\end{aligned}$$

Now use the facts that (a) for all sufficiently large $n$, $\tau_{n+1} > (\tau_n + \xi)^{\rho^*}$ for $3/2 < \rho^* < \rho$, and (b) there exists a subsequence $(S_{n_m})$, of $(S_n)$ with the property that

$$\sin(S_{n_m+1} + \beta) \ge \sin(\epsilon^*) =: \nu > 0$$

For this subsequence, defining $\delta^* := \rho^* - 3/2$,

$$-S_{N_m+1}\sin(S_{N_m+1}+\beta) + \frac{2}{3}\frac{(T_{N_m}+\xi)^3}{S_{N_m+1}} < -\nu(T_{N_m}+\xi)^{\frac{3}{2}+\delta^*} + \frac{2}{3}(T_{N_m}+\xi)^{\frac{3}{2}-\delta^*}$$

for suitably large $m$, and

$$-\nu(T_{N_m}+\xi)^{\frac{3}{2}+\delta^*} + \frac{2}{3}(T_{N_m}+\xi)^{\frac{3}{2}-\delta^*} \to -\infty \text{ as } m \to \infty.$$

So $\liminf_{t\uparrow\omega} \frac{1}{k(t)}\int_{t_0}^{t} H(k(t))\dot{k}(t)\,dt = \liminf_{\sigma\to\infty} \frac{1}{\sigma}\int_{k(t_0)}^{\sigma} H(\theta)\,d\theta = -\infty$, which by an analogous argument as in Theorem 2.2.1 and using (2.15) gives a contradiction. Therefore $k(\cdot)$ must be bounded.

Again the boundedness of $w(\cdot)$ and $y(\cdot)$ are considered, to show the solution of (2.9) exists on the interval $[t_0, \infty)$. By an analogous argument and the same $k^*$ as that in Theorem 2.2.1,

$$\langle \nabla W(x), \phi \rangle \le -\frac{1}{4}(\|w\|^2 + \|y\|^2) + (k^* + H(k))(f(w,y)\|y\| + \|y\|^2),\ \forall \phi \in F(x). \quad (2.16)$$

So for almost all $t$,

$$\frac{dW(x(t))}{dt} \le -\frac{1}{4}\int_{t_0}^{t}(\|w(t)\|^2 + \|y(t)\|^2) + \int_{t_0}^{t}(k^* + H(k(t)))\dot{k}(t)\,dt.$$

This gives

$$W(x(t)) \le W(x(t_0)) + \int_{k(t_0)}^{k(t)} (k^* + H(\theta))\,d\theta.$$

Since $k(\cdot)$ is bounded the boundedness of $w(\cdot)$ and $y(\cdot)$ is established. So by Theorem $A.3.3$ in the appendix $\omega = \infty$.

The remaining thing to prove is the solution $(w(t), y(t)) \to 0$ as $t \to \infty$. Consider the function

$$x \mapsto V^H(x) := W(x) - \int_{t_0}^{k} (k^* + H(\sigma))\, d\sigma.$$

With an analogous argument as that presented in Theorem 2.2.1

$$\langle \nabla V^H(x), \phi \rangle \le -\frac{1}{4}(\|w\|^2 + \|y\|^2), \qquad \forall \phi \in F(x).$$

So we can conclude that $(w(t), y(t)) \to (0,0)$ as $t \to \infty$, by Theorem $A.3.7$ in the appendix. □

### 2.2.4 Generating an admissible function $s(\cdot)$

In the analysis of the previous section, the importance of the alternating function $s(\cdot)$ was clear. The properties of the sequence $(\tau_n)$ were critical to the success of this function. The sequence must satisfy properties **(i)**, **(ii)** and **(iii)** of section **2.2.3**. In this section we give two methods of generating such a sequence.

#### An algorithm for generating the sequence $\tau_n$

In this subsection a method for generating the sequence $(\tau_n)$ is established. Conditions **(i)** and **(ii)** can be easily satisfied, but condition **(iii)** is not straightforward. This is due to the fact that the value $\beta$ is unknown. The method used here to overcome this uncertainty involves partitioning the interval $[0, 2\pi)$ into three distinct sections, and considering the consequences of $\beta$ being contained in each of these distinct sections.

The sequence $(\tau_n)$ is defined in the following way. Firstly define the sequence $(\mu_n)_{n\in\mathrm{N}}$ by $\mu_n := 2n\pi/3$. Fix $\rho > 3/2$ (a control parameter) and let $\tau_1 = \mu_1$. General $\tau_n$, $n > 1$, are defined as

$$\tau_n := \min\{\mu_i \mid \{i\}_3 = \{n\}_3,\ \mu_n > \tau_{n-1}^{\rho}\}. \tag{2.17}$$

As before fix $\xi \in (0, \tau_1)$ (a second control parameter) such that $\tau_2 > \tau_1 + \xi$. $s(\cdot)$ can then be defined as before with this sequence $(\tau_n)$.

The question arises: does this sequence satisfy the required properties of **(2.2.3)**? Clearly properties **(i)** and **(ii)** hold. Property **(iii)** holds by the folbwing argument. Define

$$I_0 := \begin{cases} [\frac{1}{6}\pi, \frac{5}{6}\pi), & |O| = 1 \\ [\frac{5}{6}\pi, \frac{3}{2}\pi), & |O| = -1 \end{cases}$$
$$I_1 := \begin{cases} [\frac{5}{6}\pi, \frac{3}{2}\pi), & |O| = 1 \\ [0, \frac{1}{6}\pi) \cup [\frac{3}{2}\pi, 2\pi), & |O| = -1 \end{cases}$$
$$I_2 := \begin{cases} [0, \frac{1}{6}\pi) \cup [\frac{3}{2}\pi, 2\pi), & |O| = 1 \\ [\frac{1}{6}\pi, \frac{5}{6}\pi), & |O| = -1 \end{cases}$$

Recall the definition of $(S_n)$,

$$S_n = \begin{cases} \tau_{2n+1}, & |O| = +1 \\ \tau_{2n}, & |O| = -1 \end{cases}.$$

So if

$\beta \in I_0$, the subsequence $(S_{3n-2})_{n\in\mathbb{N}}$ of $(S_n)_{n\in\mathbb{N}}$ ensures $\{S_n + \beta\}_{2\pi} \in [\frac{1}{6}\pi, \frac{5}{6}\pi]$.

$\beta \in I_1$, the subsequence $(S_{3n-1})_{n\in\mathbb{N}}$ of $(S_n)_{n\in\mathbb{N}}$ ensures $\{S_n + \beta\}_{2\pi} \in [\frac{1}{6}\pi, \frac{5}{6}\pi]$.

$\beta \in I_2$, the subsequence $(S_{3n})_{n\in\mathbb{N}}$ of $(S_n)_{n\in\mathbb{N}}$ ensures $\{S_n + \beta\}_{2\pi} \in [\frac{1}{6}\pi, \frac{5}{6}\pi]$.

So it is clear that the sequence $(\tau_n)_{n\in\mathbb{N}}$ satisfies all three of the required properties as described in the previous section.

**Another algorithm for generating the sequence $\tau_n$**

Here another sequence will be generated which satisfies the properties **(i)**, **(ii)** and **(iii)** of section **(2.2.3)**. This method will construct a sequence using the theory of uniform distributions modulo a constant. A function will be given from which the $\tau_n$ can be generated. Uniform distributions are used in this construction, see Appendix C, [41] and [25].

We proceed as follows. Firstly a theorem is proved which gives a method of generating a sequence, call it $(\mu_n)$, with certain properties. A second theorem will show that a sequence $(\tau_n)$ can be generated from two of these previously generated sequences, which satisfy the properties of section **(2.2.3)**.

**Lemma 2.2.3** *Let $(\mu_n)$ be a sequence with $\mu_1 > 1$. Let $\beta \in \mathbb{R}$, $\eta_1 > 1$, $\eta_2 > \sqrt{2}$ and $\alpha > 1$. If $(\mu_n)_{n\in\mathbb{N}}$ satisfies*

$$\mu_{n+1} > \eta_1(\mu_n)^{\eta_2}, \qquad \forall n \in \mathbb{N},$$

*then for almost all $a_1$, $a_2 \in [1/\alpha, 1]$ there exists subsequences $(\mu_{2n_l})_{l\in\mathbb{N}}$ and $(\mu_{2n_m-1})_{m\in\mathbb{N}}$, of $(\mu_{2n})_{n\in\mathbb{N}}$ and $(\mu_{2n-1})_{n\in\mathbb{N}}$ respectivly, with the property that for some $\nu > 0$*

$$\nu \le \sin(2\pi a_2 \mu_{2n_l} + \beta) \text{ and } \nu \le \sin(2\pi a_1 \mu_{2n_m-1} + \beta)$$

*for all $m$ and $l \in \mathbb{N}$.*

**Proof:** For clarity the proof will follow the stages below:

(i) Firstly the sequences $(a_1\mu_{2n-1})_{n\in\mathbb{N}}$ and $(a_2\mu_{2n})_{n\in\mathbb{N}}$ will be proven to be uniformly distributed modulo 1, for almost all $a_1$ and $a_2 \in \mathbb{R}$. The constants $a_1$ and $a_2$ will be set to viable values in the interval $[1/\alpha, 1]$.

(ii) The sequences $(2\pi a_1\mu_{2n-1} + \beta)_{n\in\mathbb{N}}$ and $(2\pi a_2\mu_{2n} + \beta)_{n\in\mathbb{N}}$ will be proven to be uniformly distributed modulo $2\pi$ for any $\beta$.

(iii) The proof will be completed.

Part (i): To prove this part of the lemma it will be shown that for some positive constants $\rho_1$ and $\rho_2$, $|\mu_{2m-1} - \mu_{2n-1}| \geq \rho_1$ and $|\mu_{2m} - \mu_{2n}| \geq \rho_2$, for all natural numbers $m \neq n$. The result then follows by Theorem $C.1.1$ given in the appendix. So let $m \neq n$ (assuming w.l.o.g. $m > n$)

$$\begin{aligned}|\mu_{2m} - \mu_{2n}| &> \mu_{2n}^{p} - \mu_{2n} \\ &> \mu_{2n}^{p-1} - 1 \\ &\geq \mu_2 - 1 =: \rho_2 > 0\end{aligned}$$

where $p = \eta_2^{2(m-n)}$. So the sequence $(a_2\mu_{2n})_{n\in\mathbb{N}}$ is uniformly distributed modulo 1, for almost all $a_2 \in \mathbb{R}$, by Theorem $C.1.1$ in the appendix. An analogous argument gives the result that the sequence $(a_1\mu_{2n-1})_{n\in\mathbb{N}}$ is uniformly distributed modulo 1, for almost all $a_1 \in \mathbb{R}$. Let $a_1$, $a_2 \in [1/\alpha, 1]$ be such that the sequences $(a_2\mu_{2n})_{n\in\mathbb{N}}$ and $(a_1\mu_{2n-1})_{n\in\mathbb{N}}$ are uniformly distributed modulo 1.

Part (ii): The sequences $(2\pi a_2\mu_{2n})_{n\in\mathbb{N}}$ and $(2\pi a_1\mu_{2n-1})_{n\in\mathbb{N}}$ are uniformly distributed modulo $2\pi$, by Proposition $C.2.1$ in the appendix. Also, for any $\beta \in \mathbb{R}$ the sequences $(2\pi a_2\mu_{2n} + \beta)_{n\in\mathbb{N}}$ and $(2\pi a_1\mu_{2n-1} + \beta)_{n\in\mathbb{N}}$ are uniformly distributed modulo $2\pi$, by Lemma $C.2.2$ in the appendix.

Part (iii): There exists an $\epsilon^* \in (0, \pi/2)$ and subsequences $(\mu_{2n_l})_{l\in\mathbb{N}} \subset (\mu_{2n})_{n\in\mathbb{N}}$ and $(\mu_{2n_m-1})_{m\in\mathbb{N}} \subset (\mu_{2n-1})_{n\in\mathbb{N}}$ (with $\mu_{2n_l} < \mu_{2n_{l+1}}$ and $\mu_{2n_m-1} < \mu_{2n_{m+1}-1}$) such that $\{2\pi a_2\mu_{2n_l} + \beta\}_{2\pi} \in [\epsilon^*, \pi - \epsilon^*)$ and $\{2\pi a_1\mu_{2n_m-1} + \beta\}_{2\pi} \in [\epsilon^*, \pi - \epsilon^*)$ for all $m$, $l \in \mathbb{N}$, by Proposition $C.2.3$ in the appendix . So with $\nu := \sin(\epsilon^*)$, $\sin(2\pi a_2\mu_{2n_l} + \beta) \geq \nu$ and $\sin(2\pi a_1\mu_{2n_m-1} + \beta) \geq \nu$, for all $m$, $l \in \mathbb{N}$, proving the lemma. □

The following theorem gives the required sequence of $\tau_n$ needed in the construction of the alternating function $s(\cdot)$.

**Theorem 2.2.4** *Let $(\mu_n)$ be a sequence satisfying the hypothesis of the previous lemma with $1 < \alpha < 2\pi$, $\eta_2 = \frac{3}{2} + \epsilon$ where $\epsilon > \ln(\alpha)/\ln(\mu_1) > 0$ and $\eta_1 = (2\pi)^{\frac{1}{2}+\delta}$ where $\delta \in (0, \epsilon - \ln(\alpha)/\ln(\mu_1))$. Let $a_1$, $a_2 \in [1/\alpha, 1]$ be such that the sequences $(a_1\mu_{2n-1})_{n\in\mathbb{N}}$ and $(a_2\mu_{2n})_{n\in\mathbb{N}}$ are uniformly* *distributed modulo 1. Define the sequence* $(\tau_n)_{n\in\mathbb{N}}$ *by*

$$\tau_{2n-1} := 2\pi a_1 \mu_{2n-1}$$
$$\tau_{2n} := 2\pi a_2 \mu_{2n}$$

*then the sequence* $(\tau_n)_{n\in\mathbb{N}}$ *satisfies the three properties in section 2.2.3.*

**Proof:** Property (i) is satisfied trivially, and property (iii) is established in the Lemma 2.2.3 above. Property (ii), namely $\tau_{n+1} > (\tau_n)^p$, where $p = \frac{3}{2} + \delta$ with $\delta > 0$ a fixed constant, can be established as follows: recall

$$\epsilon > \ln(\alpha)/\ln(\mu_1) > 0 \text{ and } \delta \in (0, \epsilon - \ln(\alpha)/\ln(\mu_1)).$$

Consider

$$\tau_{n+1} - \tau_n^p = \begin{cases} 2\pi a_1 \mu_{n+1} - (2\pi a_2 \mu_n)^p, & n \text{ even} \\ 2\pi a_2 \mu_{n+1} - (2\pi a_1 \mu_n)^p, & n \text{ odd} \end{cases}.$$

With $q = \frac{3}{2} + \epsilon$, consider the case

$$\begin{aligned} 2\pi a_1 \mu_{n+1} - (2\pi a_2 \mu_n)^p &> (2\pi)^p [a_1 (\mu_n)^q - (a_2)^p (\mu_n)^p] \\ &> \left[\frac{1}{\alpha}(\mu_n)^{q-p} - 1\right] (\mu_n)^p \\ &> \left[\frac{1}{\alpha}(\mu_n)^{q-p} - 1\right] \\ &> \frac{1}{\alpha}(\mu_n)^{\frac{\ln(\alpha)}{\ln(\mu_1)}} - 1 \end{aligned}$$

The last inequality comes from $\delta < \epsilon - \ln(\alpha)/\ln(\mu_1)$. Also note, with $\mu_n \geq \mu_1 > 1$ and $\alpha > 1$,

$$\ln(\mu_n)\ln(\alpha) \geq \ln(\mu_1)\ln(\alpha),$$

giving

$$\alpha \leq \mu_n^{\frac{\ln(\alpha)}{\ln(\mu_1)}}, \qquad \forall n \in \mathbb{N}.$$

So $2\pi a_1 \mu_{2m+1} - (2\pi a_2 \mu_{2m})^p > 0$. An analogous argument gives $2\pi a_2 \mu_{2m} - (2\pi a_1 \mu_{2m-1})^p > 0$. Hence the theorem is proved. □

It should be noted that the *two* constants $a_1$ and $a_2$ are required, since as seen in establishing the boundedness of $k(\cdot)$ in the proof of Theorem 2.2.2, the $\tau_{2n-1}$'s relate to the case $|O| = +1$ and the $\tau_{2n}$'s relate to the case $|O| = -1$. So the sequence $(\tau_{2n-1})_{n\in\mathbb{N}}$ has to be uniformly distributed modulo $2\pi$; similarly so does $(\tau_{2n})_{n\in\mathbb{N}}$. It is not sufficient for $(\tau_n)_{n\in\mathbb{N}}$ to be uniformly distributed modulo $2\pi$. For example since if you remove all the $(\tau_{2n})_{n\in\mathbb{N}}$ you can not guarantee that $(\tau_{2n-1})_{n\in\mathbb{N}}$ is uniformly distributed modulo $2\pi$.

**An example**

Here we give an example using this method of generating the $\tau_n$. Let $\mu_n = \frac{1}{2\pi}\exp(\exp(n))$, noting $\mu_1 > 1$. Let $\epsilon = 1/2$, $\alpha = 100/99$ and $\delta = \frac{1}{2}(\epsilon - \ln(\alpha)/\ln(\mu_1))$ to satisfy the theory. (Note $1/5 < \delta < 1/4$.) We must show $\mu_{n+1} > (2\pi)^{\frac{1}{2}+\delta}(\mu_n)^2$. Since $\delta < 1/4$, we can show $\mu_{n+1} > (2\pi)^{\frac{3}{4}}(\mu_n)^2$. This is done in the following way:

$$\begin{aligned}
\mu_{n+1} - (2\pi)^{\frac{3}{4}}(\mu_n)^2 &= \frac{1}{2\pi}\left(e^{e^{n+1}} - (2\pi)^{-\frac{1}{4}}e^{2e^{n}}\right) \\
&> \frac{1}{2\pi}\left(e^{e^{n+1}} - e^{2e^n}\right), \qquad \text{since } 1 > (2\pi)^{-\frac{1}{4}} \\
&= \frac{1}{2\pi}e^{2e^n}\left(e^{(e^{n+1}-2e^n)} - 1\right) > 0
\end{aligned}$$

since

$$e^n(e-2) > 0.$$

So $\mu_{n+1} > (2\pi)^{\frac{3}{4}}(\mu_n)^2$. So for almost all $a_1$, $a_2 \in [1/\alpha, 1]$, $(a_1\mu_{2n-1})_{n\in\mathrm{N}}$ and $(a_2\mu_{2n})_{n\in\mathrm{N}}$ are uniformly distributed modulo 1. Therefore the sequence $(\tau_n)_{n\in\mathrm{N}}$ defined by

$$\begin{aligned}
\tau_{2n-1} &= a_1 e^{e^{2n-1}} \\
\tau_{2n} &= a_2 e^{e^{2n}}
\end{aligned}$$

satisfies the properties required by the sequence in subsection (2.2.3), and therefore can be used in the construction of the function $s(\cdot)$.

The function $s(\cdot)$ can now be explicitly constructed. Let $\xi \in (0, \tau_2 - \tau_1)$ as required in the definition of $s(\cdot)$. Note the zeros of the function $k \mapsto \sin(\pi\ln(\ln(k/a_2))/2)$ occur at the values $k$ such that

$$\frac{1}{2}\ln\left(\ln\left(\frac{k}{a_2}\right)\right) = n \in \mathbb{N}.$$

Assuming $k > a_2 e$, the zeros occur at $k$ such that $k = a_2\exp(\exp(2n))(= \tau_{2n})$. Also the zeros of the function $k \mapsto \cos(\pi\ln(\ln(k/a_1))/2)$ occur at $k$ such that

$$\frac{1}{2}\left\{1 + \ln\left(\ln\left(\frac{k}{a_1}\right)\right)\right\} = n \in \mathbb{N}.$$

Assuming $k > a_1 e$, the zeros occur at $k$ such that $k = a_1\exp(\exp(2n-1))(= \tau_{2n-1})$. For the purposes of this example define the function sgn : $\mathbb{R} \to \{-1, +1\}$ as the following:

$$\mathrm{sgn} : k \mapsto \begin{cases} +1, & k \geq 0 \\ -1, & k < 0. \end{cases}$$

Therefore the function

$$s_1^* : k \mapsto \mathrm{sgn}(\cos(\pi\ln(\ln(k/a_1))/2))$$

is a switching function between $-1$ and $+1$, with switching points at $\tau_{2n-1}$ for all $n \in \mathbb{N}$. Similarly the function

$$s_2^* : k \mapsto \operatorname{sgn}(\sin(\pi \ln(\ln(k/a_2))/2))$$

is a switching function between $-1$ and $+1$, with switching points at $\tau_{2n}$ for all $n \in \mathbb{N}$. Therefore the function

$$s^*(k) := s_1^*(k)s_2^*(k)$$

is a discontinuous switching function between $-1$ and $+1$ with switching points $\tau_n$ for all $n \in \mathbb{N}$. The continuous function $s(\cdot)$ can be built from the discontinuous function $s^*(\cdot)$. For $\xi$ above define the piecewise linear function $\lambda_\xi : [\max\{a_1 e, a_2 e\}, \infty) \to [0,1]$ as

$$\lambda_\xi(k) := \frac{1}{4\xi}|s^*(k) - s^*(k-\xi)| \int_{k-\xi}^{k} |s^*(\tau) - s^*(\tau - \xi)|\, d\tau.$$

Note $|s^*(k) - s^*(k-\xi)|$ will have the value 2 on the set $\cup_{n\in\mathbb{N}}((\tau_{2n-1}, \tau_{2n-1} + \xi] \cup [\tau_{2n}, \tau_{2n} + \xi))$, and 0 elsewhere. We now define the function $s(\cdot)$ as

$$s(k) := s^*(k-\xi) + \lambda_\xi(k)(s^*(k) - s^*(k-\xi)).$$

and it can be verified it has the correct properties.

## 2.3 Examples

In this section some examples are given.

### 2.3.1 Example 1

Let $p = 2$ in equation (2.1). Let $M^*$ and $B^*$ of (2.1) satisfy assumption **(A1)**, and let **(A2)** be satisfied with $\gamma(v_1, v_2) = \exp(\|v_1\| + \|v_2\|)$. This is all the controller need know about the system. For the purposes of this example, with $z(t) = [z_1(t)\ z_2(t)]^T$ and $\dot{z}(t) = [\dot{z}_1(t)\ \dot{z}_2(t)]^T$, let $C_1 = I_{2\times 2}$ in the linear system $S$; i.e. $S$ is $\dot{z}(t) + z(t) = 0$. The function

$$g(t, z(t), \dot{z}(t)) = \begin{bmatrix} -\alpha_1 z_1^{\frac{1}{3}}(t) + \alpha_2 \cos(z_1(t)\dot{z}_1(t)) \\ \alpha_3 |\dot{z}_2(t)|^{\frac{1}{2}} \end{bmatrix},$$

where $\alpha_i \in \mathbb{R}$, $i \in \{1,2,3\}$ satisfies **(A2)**. In this specific example we let $\alpha_i = 1$ for all $i \in \{1,2,3\}$. Also letting

$$M^* = I \text{ and } B^* = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix},$$

satisfies **(A1)**. With this specific example the following results in Figures $(2-2)$ and $(2-3)$, are obtained.

Note that $|B^*| > 0$, so $|O| = +1$. Therefore we expect $k$ to tend to a limit where $s(k) = +1$.

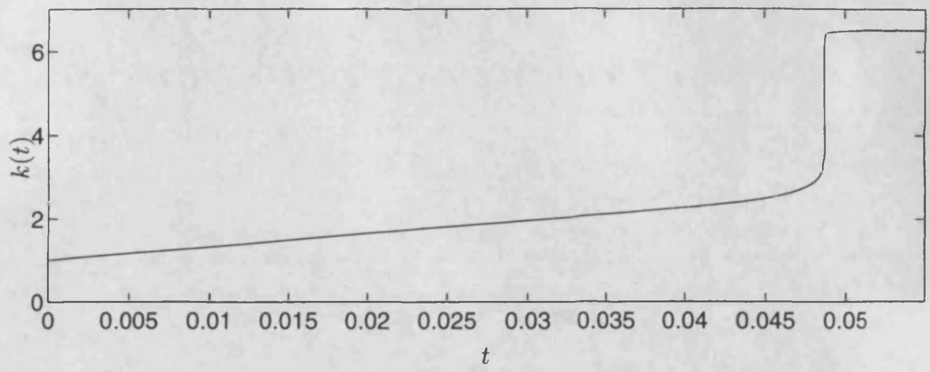


Figure 2-2: Evolution of $k$.

Since we are using the $s(k)$ generated by (2.17), we can see this is indeed the case in Figure $(2-2)$.

The Figure $(2-3)$ shows the constructed output $z(t)+\dot{z}(t)$ tends to zero as time increases. This

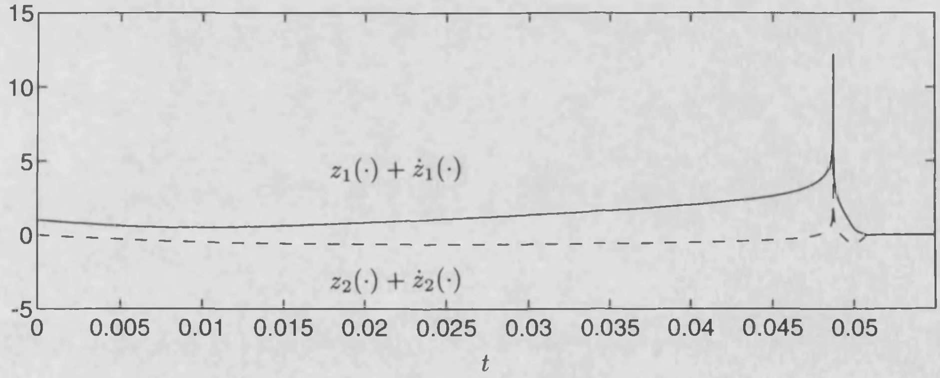


Figure 2-3: Evolution of the constructed output.

is expected by the theory. From here $z(t) \to 0$ as $t \to \infty$.

### 2.3.2 Example 2

In the previous case, $|O| = +1$. In this case we shall make $|O| = -1$, by letting

$$B^* = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}.$$

All other functions and matrices are kept the same. In this example the following results, in Figures $(2-4)$ and $(2-5)$, are obtained.

In Figure $(2-4)$ the evolution of $k$ is similar, compared to case one. The only major difference, is $k(t)$ tends to a different limit. For this limiting value of $k(t)$, $s(k) = -1$, whereas the value of $s(k)$ for the limiting value of $k(t)$ in the first case was $+1$. This is as expected by the theory

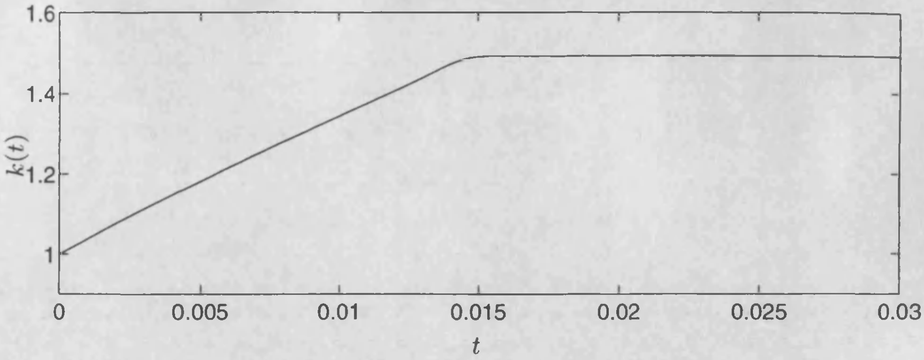


Figure 2-4: Evolution of $k$.

given earlier in this chapter.

Figure $(2-5)$ shows the evolution of the constructed output. Again this is similar to the

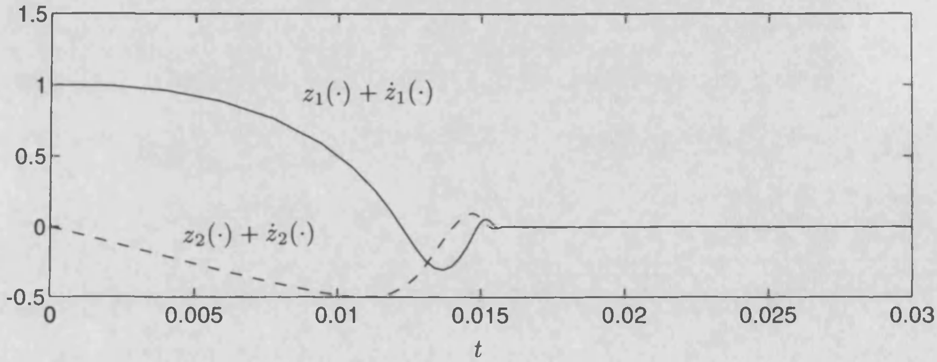


Figure 2-5: Evolution of the constructed output.

evolution of the first example, tending to zero.

## 2.4 Perturbed linear systems and output feedback

The class of system studied so far in this chapter has the full state available for feedback purposes. This is not always the case. Here we shall indicate how the control strategy of system (2.14) can be carried over to the following class of nonlinearly-perturbed linear systems:

$$\left.\begin{aligned}&\dot{\bar{x}}(t)=\bar{A}\bar{x}(t)+\bar{B}[u(t)+g(t,\bar{x}(t))]+d(t,\bar{x}(t)),\ \bar{x}(t)\in\mathbb{R}^n,\ u(t)\in\mathbb{R}^2,\ \bar{x}(t_0)=x^0\\&\bar{y}(t)=\bar{C}\bar{x}(t),\qquad \bar{y}(t)\in\mathbb{R}^k,\qquad k\geq 2.\end{aligned}\right\}\tag{2.18}$$

under the following assumptions:

**Assumptions**

**(AA1)** For some known $\bar{D} \in \mathbb{R}^{2\times k}$, the triple $(\bar{D}\bar{C}, \bar{A}, \bar{B})$ defines a minimum phase linear system of relative degree one, that is,

$$\operatorname{rank}\begin{bmatrix} sI - \bar{A} & \bar{B} \\ \bar{D}\bar{C} & 0 \end{bmatrix} = n + m, \qquad \forall s \in \bar{\mathbb{C}}_+.$$

where $\bar{\mathbb{C}}_+$ denotes the closed right half complex plane, and

$$\hat{B} := \bar{D}\bar{C}\bar{B} \in Gl(2;\mathbb{R})$$

**(AA2)** (i) For each $\bar{x} \in \mathbb{R}^n$, the function $g(\cdot, \bar{x})$ is measurable; (ii) for almost all $t \in \mathbb{R}$ $g(t, \cdot)$ is continuous; (iii) there exist a scalar $\mu_1 > 0$ and continuous function $\gamma$ such that, for almost all $t \in \mathbb{R}$,

$$\|g(t, \bar{x})\| \leq \mu_1 \gamma(\bar{C}\bar{x}), \qquad \forall \bar{x} \in \mathbb{R}^n$$

**(AA3)** (i) For all $\bar{x} \in \mathbb{R}^n$, $d(\cdot, \bar{x})$ is measurable; (ii) For almost all $t \in \mathbb{R}$, $d(t, \cdot)$ is continuous; (iii) the function $d$ is bounded in the following manner:

$$\|d(t, \bar{x})\| \leq \mu_2 \|\bar{D}\bar{C}\bar{x}\| \qquad \forall (t, \bar{x}) \in \mathbb{R} \times \mathbb{R}^n$$

where $\mu_2$ can be an unknown constant.

Notice the assumption **(AA1)** requires the existence of a known matrix $\bar{D}$ so that the resulting system, with constructed output $\bar{D}\bar{C}\bar{x} \in \mathbb{R}^2$, is minimum phase. The constructed output is needed when the number of outputs exceeds the number of inputs.

### 2.4.1 Co-ordinate transformation

In this setting where the full state is not available for feedback purposes, the following co-ordinate transformation shall be used:

From [65] and [30] if $\det(\bar{D}\bar{C}\bar{B}) \neq 0$, then the state space $\mathbb{R}^n = \mathrm{im}\bar{B} \oplus \ker\bar{D}\bar{C}$. Let $V \in \mathbb{R}^{n\times(n-m)}$ denote a basis matrix of $\ker\bar{D}\bar{C}$ (ie. the columns of $V$ are the basis vectors of $\ker\bar{D}\bar{C}$), then $U := [V, \bar{B}(\bar{D}\bar{C}\bar{B})^{-1}]$ has the inverse

$$U^{-1} = \begin{bmatrix} T \\ \bar{D}\bar{C} \end{bmatrix}, \text{ where } T := (V^T V)^{-1} V^T [I_n - \bar{B}(\bar{D}\bar{C}\bar{B})^{-1}\bar{D}\bar{C}].$$

*Sketch Proof:* Here we give a sketch proof, detailing this is indeed the inverse of $U$.

$$
\begin{aligned}
U^{-1}U &= \begin{bmatrix} TV & T\bar{B}(\bar{D}\bar{C}\bar{B})^{-1} \\ \bar{D}\bar{C}V & I \end{bmatrix} \\
&= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix},
\end{aligned}
$$

by the following argument: $\bar{D}\bar{C}V = 0$ by the construction of $V$. Also $T\bar{B}(\bar{D}\bar{C}\bar{B})^{-1} = 0$, and $TV = (V^TV)^{-1}V^TV = I$. Also

$$
\begin{aligned}
UU^{-1} &= [\bar{B}(\bar{D}\bar{C}\bar{B})^{-1}, V] \begin{bmatrix} \bar{D}\bar{C} \\ T \end{bmatrix} \\
&= (I - V(V^TV)^{-1}V^T)\bar{B}(\bar{D}\bar{C}\bar{B})^{-1}\bar{D}\bar{C} + V(V^TV)^{-1}V^T.
\end{aligned}
$$

Now $\Pi := V(V^TV)^{-1}V^T$ is a projection onto $\ker\bar{D}\bar{C}$. (For if $x \in \mathbb{R}^n$, $x = \bar{B}y + z$ for some $y \in \mathbb{R}^{n\times 2}$ and $z \in \ker\bar{D}\bar{C}$. Now $V(V^TV)^{-1}V^T\bar{B} = 0$ since $V^T$ has rows which are basis vectors of $\ker\bar{D}\bar{C}$. Also since $\bar{D}\bar{C}V = 0$, $\mathrm{Image}V(V^TV)^{-1}V^T \subset \ker\bar{D}\bar{C}$.) So for $x = \bar{B}y + z \in \mathbb{R}^n$

$$
\begin{aligned}
UU^{-1}x &= (I - V(V^TV)^{-1}V^T)\bar{B}y + V(V^TV)^{-1}V^T(\bar{B}y + z) \\
&= x,
\end{aligned}
$$

ie., $UU^{-1} = I$. □

Therefore the state space transformation

$$
\begin{bmatrix} w \\ y \end{bmatrix} = U^{-1}\bar{x} = \begin{bmatrix} T\bar{x} \\ \bar{D}\bar{C}\bar{x} \end{bmatrix}
$$

transforms (2.18) into the form

$$
\left.
\begin{aligned}
&\dot{w}(t) - L_1w(t) - L_2y(t) \in \{v \mid \|v\| \le \mu_2\|T\|\|y\|\} \\
&\dot{y}(t) + L_3w(t) + L_4y(t) - \hat{B}u \in \mu^*\mathcal{F}(\bar{y})
\end{aligned}
\right\} \tag{2.19}
$$

where $\mathcal{F}(\bar{y}) = f(\bar{y})\mathbb{B}_2$, $\mu^* = \mu_1\|\hat{B}\| + \mu_2\|\bar{D}\bar{C}\|$ and $f(\bar{y}) := \gamma(\bar{y}) + \|\bar{y}\|$ This system, with the absence of the set in the dynamics of $w$ can be identified with (2.6) by setting $M = I$, $B = -\hat{B}$ and $C_{p-1} = -L_4$.

By the minimum-phase condition of **(AA1)** $\sigma(L_1) \subset \mathbb{C}_-$ by the observation, for $s \in \bar{\mathbb{C}}_+$

$$
\begin{aligned}
0 \neq \det & \begin{bmatrix} sI - \bar{A} & \bar{B} \\ \bar{D}\bar{C} & 0 \end{bmatrix} \\
= \det & \begin{bmatrix} sI - L_1 & -L_2 & 0 \\ L_3 & sI + L_4 & \bar{D}\bar{C}\bar{B} \\ 0 & I & 0 \end{bmatrix} \\
= & \det[sI - L_1] \det[\bar{D}\bar{C}\bar{B}].
\end{aligned} \tag{2.20}
$$

### 2.4.2 Adaptive output feedback strategy

Under this transformation the system (2.18) is presented in a similar format to that of the transformed system (2.6) on which the analysis of previous sections was done. This results in a similar control structure as previously encountered. Let $\mathcal{O}_+(\cdot)$, $\mathcal{S}(\cdot)$, $s(\cdot)$ and $\psi(\cdot)$ be as before. Our adaptive feedback strategy is

$$u(t) \in \bar{\Psi}(\bar{y}(t), k(t)), \qquad \dot{k}(t) = f(\bar{y}(t))\|\bar{D}\bar{y}(t)\| + \|\bar{D}\bar{y}(t)\|^2, \qquad k(0) = k^0$$

where

$$\bar{\Psi}(\bar{y}, k) := k^2[f(\bar{y}) + \|\bar{D}\bar{y}\|]\mathcal{O}_+(k)\mathcal{S}(s(k))\psi(\bar{D}\bar{y})$$

An analogous argument to that in previous sections of this chapter proves this strategy is a universal stabiliser for (2.18), with the following observation.

Where $\langle \nabla W(w), \phi \rangle$, $\phi \in F(x)$ was considered in the proof of Theorem 2.2.2, a new term $\langle Pw, v \rangle$ will appear, where $v \in (\mu_2 \|T\| \|y\|)\mathbb{B}$. Amending the squared term in $k^*$ to $\|PL_2\| + \|Q^{-1}L_3\| + \mu_2\|T\|\|P\|$, will allow the proof to proceed as before.

# Chapter 3

# Stabilisation of nonlinearly perturbed linear systems: the multi-input case

## 3.1 Introduction

In the previous chapter a class of two-input systems was studied. The polar decomposition of the $2 \times 2$ invertible matrices was exploited in constructing Liapunov functions to show that, by feedback controls with an adaptive gain, a target can be made globally attractive. In this chapter we again study a similar system with the added generality of $m$-inputs.

As seen in the previous chapter, the problem of multi-inputs is far from straightforward. Theory already established uses the idea of the existence of a spectrum unmixing sets of finite cardinality. The existence of such unmixing sets for the invertible, $m \times m$, real matrices are conjectured in [13] and proved by Mårtensson [49] (see also [50]). The proof given by Mårtensson [49] is non-constructive in nature. By this we mean a method is not given for constructing a finite unmixing set for given $m$; existence is just established. Since the unmixing set is used in the control, it must be known *a priori*. In some of the work in this area, where a finite unmixing set strategy is used, see [33], [73] and [70], the knowledge of a finite unmixing set is often among the assumptions imposed on the system being studied. Other work, see [31], avoids some analytical complications by restricting the class of high-frequency gain matrices in the following manner: the spectrum of the matrix is entirely within the open left half complex plane *or* entirely within the open right half complex plane. Clearly the latter case is very restrictive on the class of system permitted. With regard to the former case, it would be more practical to impose conditions on the permitted class of high-frequency gain matrix, which in turn would lead to a known finite unmixing set. In this chapter it is this compromise that is sought.

The class of invertible, high-frequency gain matrices to be considered in this chapter will be *essentially column diagonally dominant* matrices. By 'essentially' column diagonally dominant, we mean the matrix can be transformed into a column diagonally dominant matrix by swapping its columns. The definition will be properly given in the chapter. In a later chapter, the more standard idea of diagonal dominance with respect to rows will be considered. Therefore for clarity, row or column shall be specified when referring to strict diagonal dominance. By imposing this extra assumption on the high-frequency gain matrix, a known, finite unmixing set will be explicitly given. This is the basis of this chapter; using this known finite unmixing set for the given class of system. The ideas found here will be extended and used in the systems with stronger nonlinearities which will be studied in later chapters of this thesis.

## 3.2 A class of systems

The system that will be studied in this chapter will be the general $m$-input case of the system encountered in the previous chapter. A further restriction on the high-frequency gain matrix will be made. Explicitly the system to be studied in this chapter will be the following:

$$\begin{aligned} &M^{*}z^{(p)}(t)+B^{*}u(t)=g(t,z(t),\dot{z}(t),\ldots,z^{(p-1)}(t)),\\ &(z(0),\dot{z}(0),\ldots,z^{(p-1)}(0))\in\mathbb{R}^m\times\cdots\times\mathbb{R}^m \end{aligned} \tag{3.1}$$

where $z(t)$, $u(t)\in\mathbb{R}^m$. $M^*$, $B^*$ and $g$ (assumed measurable in $t$ and continuous in $(z,\dot{z},\ldots,z^{(p-1)})$) are uncertain. The following structural assumptions are imposed:

**Structural Assumptions:**

**(A1):** $M^*$, $B^*\in\mathcal{GL}(m;\mathbb{R})$, with ${M^*}^{-1}B^*$ invertible and essentially column diagonally dominant. (The definition of this is given later in this chapter).

**(A2):** $g$ is bounded modulo an unknown scalar multiplier $\mu>0$, by a known continuous function of the state, $\gamma$, in the sense that, for almost all $t\in\mathbb{R}$

$$\|g(t,v_1,v_2,\cdots,v_p)\|\le\mu\gamma(v_1,\cdots,v_p),\qquad\forall\,(v_1,\cdots,v_p)\in\mathbb{R}^m\times\cdots\mathbb{R}^m.$$

Note this states that $g$ is uniformly bounded in explicit $t$ dependence. An example of a typical $g$ can be found in the previous chapter.

Write $M=\mu^{-1}M^*$ and $B=\mu^{-1}B^*$, and define the set-valued map

$$\mathcal{Z}\,:\,(v_1,\cdots,v_p)\mapsto\gamma(v_1,\cdots,v_p)\overline{\mathbb{B}}_m$$

From the argument given in the previous chapter $\mathcal{Z}$ is a known, continuous set-valued map from $\mathbb{R}^{mp}$ to the non-empty, convex and compact subsets of $\mathbb{R}^m$. The system (3.1) can now be embedded in the differential inclusion:

$$\begin{aligned} &Mz^{(p)}(t) + Bu(t) \in \mathcal{Z}(z(t), \dot{z}(t), \dots, z^{(p-1)}(t)), \\ &(z(0), \dot{z}(0), \dots, z^{(p-1)}(0)) \in \mathbb{R}^m \times \cdots \times \mathbb{R}^m \end{aligned} \tag{3.2}$$

where $z(t)$, $u(t) \in \mathbb{R}^m$. Thus the analytical framework is once again that of differential inclusions [3] and [18] (which again is also appropriate to the essentially discontinuous nature of the feedback strategy).

The aim of this chapter will be to construct an adaptive feedback control that will render the origin globally attractive. For the class of high-frequency gain matrix permitted, the control will involve a strategy that cycles through the elements of the known, finite unmixing set of matrices constructed below.

### 3.2.1 A co-ordinate transformation

Here a useful co-ordinate transformation is made, similar to that of the previous chapter. Such transformations can be found in references [70] and [73]. Let $C_i \in \mathbb{R}^{m\times m}$, $i = 1, 2, \cdots, p-1$ be such that all the poles of the linear system

$$S\,:\, z^{(p-1)}(t) + C_{p-1}z^{(p-2)}(t) + \cdots + C_2\dot{z}(t) + C_1 z(t) = 0$$

lie in the open left half plane $\mathbb{C}_-$. Define the transformation $T$ as

$$T\,:\,(z(t), \dot{z}(t), \cdots, z^{(p-2)}(t)) \mapsto (w(t), y(t)),$$

where $w(\cdot)$ and $y(\cdot)$ are defined as in (2.4) and (2.5). This transformation takes (3.2) into the form

$$\begin{aligned} &\dot{w}(t) = L_1 w(t) + L_2 y(t) \\ &M[\dot{y}(t) + L_3 w(t) - C_{p-1}y(t)] + Bu(t) \in \mathcal{F}(w(t), y(t)) \\ &(w(0), y(0)) \in \mathbb{R}^{m(p-1)} \times \mathbb{R}^m \end{aligned} \tag{3.3}$$

where $\mathcal{F} := \mathcal{Z} o T^{-1}$ and $L_1$, $L_2$ and $L_3$ are linear. Again $\sigma(L_1) \subset \mathbb{C}_-$, which plays an important role in the analysis, see section **(2.2.1)** for details.

## 3.3 Essentially diagonally dominant matrices

Here we shall construct a finite unmixing set for a class of matrices. In this construction process, Gerschgorin's theorem on matrices, [5], will be used. This theorem is repeated here since the ideas involved lead to a greater understanding of the unmixing set to be used in the control. The reader might find [20], [76] and [80] as useful references during this section.

### 3.3.1 Gerschgorin's theorem

The key to the construction of the finite unmixing set, for the class of systems, is Gerschgorin's Disc theorem. This theorem can be found in many texts, such as [5] for example.

**Notation:** Let $\mathcal{M}_n$ be the set of real $n \times n$ matrices and $\mathcal{M}_n^{\mathbb{C}}$ be the set of complex $n \times n$ matrices.

**Notation:** Given a matrix $A \in \mathcal{M}_n^{\mathbb{C}}$ and $1 \le j \le n$, define

$$\mathcal{D}_c(A,j) := \left\{ \lambda \in \mathbb{C} \;\middle|\; |\lambda - a_{jj}| \le \sum_{i=1,\, i\neq j}^{n} |a_{ij}| \right\}.$$

This is precisely the closed disc centered at $a_{jj}$ in the complex plane with radius equal to the sum of the absolute values of the other entries in the $j^{\text{th}}$ column. This will be referred to as the $j^{\text{th}}$ *column disk of the matrix* $A$, or the $j^{\text{th}}$ *column disk* when the matrix is implicit.

**Notation:** Given a matrix $A \in \mathcal{M}_n^{\mathbb{C}}$ and $1 \le i \le n$, define

$$\mathcal{D}_r(A,i) := \left\{ \lambda \in \mathbb{C} \;\middle|\; |\lambda - a_{ii}| \le \sum_{j=1,\, j\neq i}^{n} |a_{ij}| \right\}.$$

This is precisely the disc centered at $a_{ii}$ with radius equal to the sum of the absolute values of the other entries in the $i^{\text{th}}$ row. This will be referred to as the $i^{\text{th}}$ *row disk of the matrix* $A$, or the $i^{\text{th}}$ *row disk* when the matrix is implicit.

**Notation:** The union of all column (respectively, row) discs of $A$ will be denoted by $\mathcal{D}_c(A)$ (respectively, $\mathcal{D}_r(A)$), ie.

$$\mathcal{D}_c(A) := \cup_{j=1}^{n} \mathcal{D}_c(A,j), \quad \text{and} \quad \mathcal{D}_r(A) := \cup_{i=1}^{n} \mathcal{D}_r(A,i).$$

Gerschgorin's theorem can now be stated as follows:

**Theorem 3.3.1** *Let $A \in \mathcal{M}_n$, then $\sigma(A) \subset \mathcal{D}_c(A) \cap \mathcal{D}_r(A)$.*

**Proof:** This proof can be found in [5] or many new basic texts in matrix theory, but is repeated here for completeness. Let $\lambda$ be an eigenvalue of the matrix $A$, and $x = [x_1 \ldots x_n] \neq 0$ be an associated eigenvector. Let $i$ be such that $|x_i| \ge |x_l|$ for all $l$, $1 \le l \le n$. Since $Ax = \lambda x$,

$$\sum_{j=1}^{n} a_{ij} x_j = \lambda x_i.$$

whence

$$|a_{ii} - \lambda|\,|x_i| = \left| \sum_{j=1,\, j\neq i}^{n} a_{ij} x_j \right| \leq \sum_{j=1,\, j\neq i}^{n} |a_{ij}|\,|x_i|.$$

Dividing both sides by $|x_i|$ (non-zero since $x$ is an eigenvector) gives the result $\lambda \in \mathcal{D}_r(A, i)$, and so $\sigma(A) \subset \mathcal{D}_r(A)$. The result $\sigma(A) \subset \mathcal{D}_c(A)$ is proved in a similar way (noting that the spectrum of $A$ coincides with that of its transpose). This concludes the proof. □

In this chapter the matrices have real elements since in practical situations in control theory this is the case. This means the centers of the Gerschgorin discs are on the real axis, since the centers are the $a_{ii}$.

### 3.3.2 An unmixing set for invertible column diagonal dominant matrices

In this subsection a finite unmixing set is constructed for the class of real, invertible matrices with the property of column diagonal dominance. The concept of reflecting Gerschgorin discs in the imaginary axis will play a central role in the analysis. Firstly some notation and definitions will be found to be useful.

**Notation:** Let $\mathcal{I}_n$ be the set of diagonal $n \times n$ matrices with plus or minus one in the entries, ie.

$$\mathcal{I}_n := \{M \in \mathcal{M}_n \mid M = \operatorname{diag}\{d_1, \dots d_n\},\ d_i \in \{-1, 1\}\}$$

**Definition:** A matrix $A := [a_{ij}] \in \mathcal{M}_n^{\mathbb{C}}$ is *row diagonally dominant* if

$$\sum_{j=1,\, j\neq i}^{n} |a_{ij}| \leq |a_{ii}|$$

for all $i$, $1 \leq i \leq n$. $A$ is *column diagonally dominant* if

$$\sum_{i=1,\, i\neq j}^{n} |a_{ij}| \leq |a_{jj}|.$$

for all $j$, $1 \leq j \leq n$.

These definitions lead to the following established lemma. It should be noted that the proof is completely constructive in nature, which is the key idea in the chapter.

**Lemma 3.3.2** *The set $\mathcal{I}_n$ is a finite unmixing set for the class of real, invertible matrices which are column diagonally dominant.*

**Proof:** Let $A = [a_{ij}] \in \mathcal{M}_n$ be a of column diagonally dominant matrix. Define the discontinuous function sgn : $\mathbb{R} \to \{-1, 0, 1\}$ by

$$\mathrm{sgn}(x) := \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

Note that $a_{ii} \neq 0$ for all $i$, since $A$ is invertible and column diagonally dominant. Define the matrix $B = \mathrm{diag}(\mathrm{sgn}(a_{11}), \mathrm{sgn}(a_{22}), \ldots \mathrm{sgn}(a_{nn})) \in \mathcal{I}_n$. The claim is $B \in \mathcal{I}_n$ unmixes $A$. This is shown by the following argument:

$$\begin{aligned} \mathcal{D}_c(AB, j) &= \left\{ z \in \mathbb{C} \,\middle|\, |z - \mathrm{sgn}(a_{jj})a_{jj}| \leq \sum_{i=1,\, i \neq j}^{n} |\mathrm{sgn}(a_{jj})a_{ij}| \right\} \\ &= \left\{ z \in \mathbb{C} \,\middle|\, |z - |a_{jj}|| \leq \sum_{i=1,\, i \neq j}^{n} |a_{ij}| \right\} \subset \mathbb{C}_+ \end{aligned}$$

Using Gerschgorin's Theorem 3.3.1, with $\lambda$ an eigenvalue of $AB$,

$$\lambda \in \cup_{j=1}^{n} \mathcal{D}_c(AB, j) = \mathcal{D}_c(AB) \subset \mathbb{C}_+$$

This completes the proof. □

Due to the constructive nature of the proof, given a real column diagonally dominant matrix, the unmixing matrix of $\mathcal{I}_n$ can be easily found.

**Example:** From the proof, the member of $\mathcal{I}_n$ which unmixes

$$\begin{bmatrix} 3 & -2 & 3 \\ 1 & -5 & -2 \\ 1 & 1 & 6 \end{bmatrix} \text{ is the matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

### 3.3.3 Extending the class of matrices

With these basic ideas the class of matrices to be unmixed can be extended with minimal difficulty. The class of real, $n \times n$ matrices permitted will be any invertible matrix that can be transformed to a column diagonally dominant matrix under any number of column swaps, see [80]. The following notation is useful:

**Notation:** Let $e_i$ be the column vector with 1 in the $i^{\text{th}}$ entry, and zeros elsewhere.

**Notation:** Let $E_n$ be the set of real $n \times n$ matrices which swap the columns of pre-multiplying matrices, ie.

$$E_n := \{M = [v_1, \ldots v_n] \in \mathcal{M}_n \,|\, v_i \in \{e_1, \ldots e_n\}, \det(M) \neq 0\}$$

Note that the size of the set $E_n$ is finite, in fact $|E_n| = n!$.

**Notation:** Using the definitions above, define the following set

$$\mathcal{I}_n^E := \{M \in \mathcal{M}_n \mid M = GA,\ G \in E_n,\ A \in \mathcal{I}_n\}$$

Members of this set are members of $\mathcal{I}_n$ with possibly its rows swapped. Note that the size of the set $\mathcal{I}_n^E$ is finite, in fact $|\mathcal{I}_n^E| = 2^n n!$.

**Definition:** Let $M \in \mathcal{M}_n$. The matrix $M$ is essentially column diagonally dominant if there exist $G \in E_n$ such that $MG$ is column diagonally dominant.

The latter definition states a matrix $M$ is essentially column diagonally dominant if it can be made into a column diagonally dominant matrix by swapping its columns.

Using these definitions we can now give the proof of the existence a known unmixing set for essentially column diagonally dominant matrices.

**Lemma 3.3.3** *The finite set $\mathcal{I}_n^E$ is an unmixing set for the class of invertible $n \times n$ essentially column diagonally dominant matrices.*

**Proof:** The proof is simple, given the discussion above. Let $M$ be an invertible essentially diagonally dominant matrix. Therefore there exists $G \in E_n$ such that $MG$ is column diagonally dominant and invertible. From the lemma given in the previous subsection, there exist a $U \in \mathcal{I}_n$ such that $\sigma(MGU) \subset \mathbb{C}_+$. So $GU$ is the unmixing matrix in $\mathcal{I}_n^E$ of $M$. □

## 3.4 A universal control

Given the set $\mathcal{I}_n^E$ of the previous section, a universal control can be constructed that stabilises the class of systems (3.2). The form of the control will be similar to that of the previous chapter, due to the systems having many similarities. The main difference in the system of this chapter, compared with that of the previous chapter, is that it is a multi-input system of general input dimension $m$. Whereas the control in the previous chapter involved a method of cycling through the special 2-dimensional rotation matrices and their counterpart with determinant of $-1$, the control here reverts to a more conventional cycling strategy as found in [70], [73] and [50]. However, the cycling strategy found in [70] and [73] will be improved upon by removing one of the generating sequences.

### 3.4.1 Cycling strategy

In such works as [49], [73] and [70], sequences with special properties were used in the construction of a cycling strategy between elements of a finite set of unmixing matrices. As seen in [70] and [73] two sequences were needed; here only one will be needed. Define the sequence $(\tau_n)$, $\tau_{n-1} < \tau_n$ such that

$$\lim_{n\to\infty} \tau_n = \infty \quad \text{and} \quad \lim_{n\to\infty} \tau_{n-1}/\tau_n = 0.$$

A sequence so generated is unbounded and the separation between successive elements increases in the way determined by the latter property. Next, order the $r := 2^n n!$ elements of the set $I_n^E$, and relabel them as $K_1$ to $K_r$. The following notation is given:

**Notation:** For $q \in \mathbb{N}_0$ and $i \in \{1,\ldots,r\}$, define the sets

$$T_i^q := (\tau_{rq+i}, \tau_{rq+i+1}] \quad \text{and} \quad T_i := \cup_{q\in\mathbb{N}_0} T_i^q$$

**Notation:** For $q \in \mathbb{N}_0$ and $i \in \{1,\ldots,r\}$, let $\xi_i^q : T_i^q \to [0,1]$ be the continuous function

$$\xi_i^q(s) = (s - \tau_{rq+i})/(\tau_{rq+i+1} - \tau_{rq+i})$$

Note

$$\xi_i^q(\tau_{rq+i}) = 0 \quad \text{and} \quad \xi_i^q(\tau_{rq+i+1}) = 1.$$

Also define the function $\xi_i : T_i \to \mathbb{R}$ as

$$\xi_i(s) := \xi_i^q(s), \qquad s \in T_i^q$$

A cycling strategy between the elements of this finite unmixing set can now be defined as

$$K(s) := \begin{cases} K_1, & s \in (\infty, \tau_1] \\ \ldots & \ldots \\ K_i + \xi_i(s)(K_{i+1} - K_i), & s \in T_i,\ i \in \{1,2,\ldots,r-1\} \\ \ldots & \ldots \\ K_r + \xi_r(s)(K_1 - K_r), & s \in T_r \end{cases} \tag{3.4}$$

This function is continuous, cycling between the ordered elements of the finite set $\mathcal{I}_n^E$. Once the last element of $\mathcal{I}_n^E$ is reached, the function then continuously moves from the last to the first element. Also note that $K(s) \in \text{co}\{K_1, \ldots K_r\}$ for all $s$.

### 3.4.2 The universal control

The control for this system is similar in nature to the control (2.14). The main difference is the cycling strategy in the control; the one used in this chapter will use the cycling strategy $K(\cdot)$

between the matrices $K_i$. The control strategy can now formally be given as

$$\begin{aligned} u(t) &= -k(t)[f(w(t), y(t)) + \|y(t)\|]\|y(t)\|^{-1}K(k(t))y(t) \\ \dot{k}(t) &= f(w(t), y(t))\|y(t)\| + \|y(t)\|^2, \qquad k(0) \in \mathbb{R}_+ \end{aligned} \tag{3.5}$$

where $f : (w, y) \mapsto \max\{\|\phi\| \,|\, \phi \in \mathcal{F}(w, y)\}$ is as before. This function is well defined and continuous. However, the control $u$ has a discontinuity at the point $y = 0$, and could cause analytic difficulties. Again the technique presented in [70] and [73], which expresses the control, and in turn the feedback system, in a set inclusion form. This is done by reinterpreting the control in the following way. Let $x = (w, y, k)$,

$$\begin{aligned} u(t) \in \Psi(x(t)) &:= -k(t)[f(w(t), y(t)) + \|y(t)\|]K(k(t))\psi(y(t)) \\ \dot{k}(t) &= f(w(t), y(t))\|y(t)\| + \|y(t)\|^2, \qquad k(0) \in \mathbb{R}_+ \end{aligned} \tag{3.6}$$

where $\psi(\cdot)$ is defined as in (2.7).

## 3.5 Stability analysis

Using the set-valued control of the previous subsection, the transformed system (3.3) can be embedded in the following differential inclusion. Let $x = (w, y, k) \in \mathbb{R}^{m(p-1)} \times \mathbb{R}^m \times \mathbb{R}$. Define the function $F : \mathbb{R}^{mp+1} \to 2^{\mathbb{R}^{mp+1}}$ as

$$F(x) := F_1(x) \times F_2(x) \times F_3(x) \tag{3.7}$$

where $F_i$, $i \in \{1, 2, 3\}$, are defined by

$$\begin{aligned} F_1(x) &:= \{L_1 w + L_2 y\} \\ F_2(x) &:= \{M^{-1}[\phi - Bu] - L_3 w + C_{p-1} y \,|\, \phi \in \mathcal{F}(w, y),\, u \in \Psi(x)\} \\ F_3(x) &:= \{\|y\|^2 + f(w, y)\|y\|\} \end{aligned} \tag{3.8}$$

The system (3.3) can now be embedded in the inclusion

$$\dot{x}(t) \in F(x(t)), \qquad x(t_0) = x^0. \tag{3.9}$$

To guarantee a solution of this differential inclusion, it is sufficient to check the following conditions: the function $F(\cdot)$ is an upper-semicontinuous set-valued map with non-empty, convex and compact values. Clearly the sets $F_1(x)$ and $F_3(x)$ are convex and compact, since they are singleton sets. $F_2(x)$ is convex and compact since the values of $\mathcal{F}(w, y)$ and $\Psi(x)$ have these properties. The upper-semicontinuity of the singleton-valued maps $F_1(\cdot)$ and $F_3(\cdot)$ is trivial. The upper-semicontinuity of the set-valued map $F_2(\cdot)$ is established by the use of Lemma $A.2.5$ and Lemma $A.2.6$ in the appendix, by noting $F_2(\cdot)$ is composed of the sum and multiplication of upper-semicontinuous set-valued maps and continuous valued functions. Lemma $A.2.2$ in the appendix gives the upper-semicontinuity of the the set-valued map $F(\cdot)$. So a solution to the initial-value problem, not necessarily a unique solution, exists on a maximal interval $[t_0, \omega)$ by [3, Theorem 2.1.3], see Theorem $A.3.1$ and Theorem $A.3.2$. Let $x : [t_0, \omega) \to \mathbb{R}^{mp+1}$ be such a solution.

Before we proceed with the main stability theorem, which establishes the global attractivity of the origin, some useful inequalities are established. Within the analysis of the main theorem, the inner-product $-\langle y, QDK(s)y\rangle$, plays a prominent role; $K(\cdot)$ is the function met in (3.4), $Q$ is a positive definite, symmetric matrix and $D \in \mathcal{M}_m$ is an essentially column diagonally dominant invertible matrix. A bound on this term is considered in the following lemma.

**Lemma 3.5.1** *Let $D \in \mathcal{M}_m$ be invertible and essentially column diagonally dominant, let $K(\cdot)$ be the function defined in (3.4), then there exists $\epsilon \in (0,1)$, $j \in \{1,\ldots,r\}$ and $Q \in \mathcal{M}_m$, a positive definite, symmetric matrix such that for*

$$I_{\epsilon,j} := \cup_{q\in\mathbb{N}_0}[\tau_{rq+j} + \epsilon(\tau_{rq+j-1} - \tau_{rq+j}), \tau_{rq+j} + \epsilon(\tau_{rq+j+1} - \tau_{rq+j})],$$

$$s \in I_{\epsilon,j} \implies -\langle y, QDK(s)y\rangle \le -\frac{1}{4}\|y\|^2, \qquad \forall y \in \mathbb{R}^m$$

*and*

$$s \notin I_{\epsilon,j} \implies -\langle y, QDK(s)y\rangle \le \frac{1}{4}\beta\|y\|^2, \qquad \forall y \in \mathbb{R}^m$$

*for some $\beta > 0$.*

**Proof:** $K(\cdot)$ is a cycling function between the $r$ elements of the set $\mathcal{I}_n^E$. There exists a $j \in \{1,2,\ldots r\}$ such that $\sigma(DK_j) \subset \mathbb{C}_+$ by Lemma 3.3.3. By Liapunov theory, there exists a positive-definite, symmetric matrix $Q$, such that

$$QDK_j + (QDK_j)^T = I$$

Using the notation of section **(3.4)**, for $s \in T_{j-1}$

$$K(s) = K_j + (\xi_{j-1}(s) - 1)(K_j - K_{j-1}).$$

So

$$QDK(s) + [QDK(s)]^T = \begin{cases} I + (\xi_{j-1}(s) - 1)\{QD(K_j - K_{j-1}) + [QD(K_j - K_{j-1})]^T\} & s \in T_{j-1} \\ I + \xi_j(s)\{QD(K_{j+1} - K_j) + [QD(K_{j+1} - K_j)]^T\} & s \in T_j \end{cases}$$

Choose $\epsilon \in (0, \min\{1/(8\|QD\|), 1\})$, for $s \in I_{\epsilon,j}$

$$-\langle y, [QDK(s) + (QDK(s))^T]y\rangle \le -\tfrac{1}{2}\|y\|^2,$$

since $\|K_p - K_n\| \le 2$ for $p, n \in \{1, \dots, r\}$ ($K_p^T K_p = I$ for all $1 \le p \le r$). Clearly, $s \in \mathbb{R}$,

$$-\langle y, [QDK(s) + (QDK(s))^T]y\rangle \le 2\|QD\|\|K(s)\|\|y\|^2 \le \frac{1}{2}\beta\|y\|^2$$

where $\beta = 8\|QD\|$. Now recall for $M \in \mathcal{M}_m$, $\langle y, My\rangle = \langle y, (M + M^T)y\rangle/2$ for all $y$ which gives the result. □

The next lemma gives the result concerning a discontinuous function that mimics an important property of Nussbaum functions.

**Lemma 3.5.2** *Let $k^0 \in \mathbb{R}$, $(\tau_n)$ be an increasing sequence such that*

$$\lim_{n\to\infty} \tau_n = \infty \text{ and } \lim_{n\to\infty} \tau_{n-1}/\tau_n = 0,$$

*let $\alpha$, $\gamma > 0$, $0 < \epsilon < 1$, define $I_{\epsilon,j}$ as in the previous lemma, let $\nu : \mathbb{R} \to \{-\alpha\gamma/2, \alpha/2\}$ be defined as*

$$\nu(k) := \begin{cases} \alpha/2, & k \in I_{\epsilon,j} \\ -\alpha\gamma/2, & k \notin I_{\epsilon,j} \end{cases}$$

*then*

$$\limsup_{k\to\infty} \frac{1}{k}\int_{k^0}^{k} k\nu(k)\,dk = \infty.$$

**Proof:** For $q \in \mathbb{N}_0$, define $\delta_q := \tau_{rq+j} + \epsilon(\tau_{rq+j+1} - \tau_{rq+j})$ and $\rho_q := \tau_{rq+j} + \epsilon(\tau_{rq+j-1} - \tau_{rq+j})$. Consider the integral

$$\frac{1}{\delta_N}\int_{k^0}^{\delta_N} k\nu(k)\,dk = a + \frac{1}{\delta_N}\sum_{q=k^*}^{N}\left(\int_{\rho_q}^{\delta_q}\frac{\alpha}{2}k\,dk - \int_{\delta_{q-1}}^{\rho_q}\frac{\alpha\gamma}{2}k\,dk\right)$$

where $k^* = \min\{n \in \mathbb{N} \,|\, \delta_{n-1} \ge k^0\}$ and $a = \frac{1}{\delta_N}\int_{k^0}^{\delta_{k^*-1}} k\nu(k)\,dk > -\infty$. This gives

$$\begin{aligned}\frac{1}{\delta_N}\int_{k^0}^{\delta_N} k\nu(k)\,dk = a + \frac{\alpha}{4\delta_N}\sum_{q=k^*}^{N-1}[\delta_q^2 - (1+\gamma)\rho_q^2 + \gamma\delta_{q-1}^2] + \\ \frac{\alpha}{4\delta_N}[\delta_N^2 - (1+\gamma)\rho_N^2 + \gamma\delta_{N-1}^2] \qquad (3.10)\end{aligned}$$

The second term on the right hand side is bounded below uniformly in $N$ by the following argument: for all $q \in \mathbb{N}$,

$$\delta_q^2 - (1+\gamma)\rho_q^2 = (\delta_q + \sqrt{(1+\gamma)}\rho_q)(\delta_q - \sqrt{(1+\gamma)}\rho_q).$$

By the properties of the sequence $(\tau_n)$, $(\delta_q - \sqrt{(1+\gamma)}\rho_q) \to \infty$ as $q \to \infty$, and therefore is greater than a positive constant, $c^* > 0$ say, for all $q \geq N^*$. This gives the result for the second term.

The final term of the right hand side of (3.10) tends to infinity as $N \to \infty$, since

$$\begin{aligned}\frac{\alpha}{\delta_N}[\delta_N^2 - (1+\gamma)\rho_N^2] &= \frac{\alpha}{4}\left(1 + \sqrt{(1+\gamma)}\frac{\rho_N}{\delta_N}\right)\left(\delta_N - \sqrt{(1+\gamma)}\rho_N\right) \\ &> \frac{\alpha}{4}\left(\delta_N - \sqrt{(1+\gamma)}\rho_N\right) \to \infty.\end{aligned}$$

□

We now turn to the main proof of this chapter. This gives the properties of the solutions of the differential inclusion (3.9). It establishes (i) the maximal interval of existence is the infinite interval $[t_0, \infty)$, (ii) the adaptive gain $k(\cdot)$ tends to a finite limit and (iii) the origin is globally attractive with respect to the solutions $w(t)$ and $y(t)$. These results are given in the following theorem:

**Theorem 3.5.3** *Let $x = (w, y, k) : [t_0, \omega) \to \mathbb{R}^{mp+1}$ be a solution of the initial-value problem (3.9), on the maximal interval of existence, then*

*(i)* $\omega = \infty$,

*(ii)* $\lim_{t\to\infty} k(t)$ *exists and is finite,*

*(iii)* $\lim_{t\to\infty} \|(w(t), y(t))\| = 0$.

**Proof:** The proof proceeds in a similar manner as its counterpart in the second chapter, but taking care with the analytic difficulties posed by the introduction of general $m$-inputs. Let $Q$ be the positive-definite symmetric matrix of Lemma 3.5.1 and $P$ be a positive definite symmetric matrix such that

$$PL_1 + (PL_1)^T = -I.$$

Define the functions

$$W_1 : w \mapsto \frac{1}{2}\langle w, Pw\rangle \text{ and } W_2 : y \mapsto \frac{1}{2}\langle y, Qy\rangle.$$

Since $\dot{w}(t) = L_1 w(t) + L_2 y(t)$ with $\sigma(L_1) \subset \mathbb{C}_-$, from Lemma $D.1.1$ in the appendix there exists constants $c_0$ and $c_1$ such that for all $t \in [t_0, \omega)$,

$$\begin{aligned}\int_{t_0}^{t} \|y(s)\|\|w(s)\|\, ds &\leq c_0\|w(t_0)\|^2 + c_1\int_{t_0}^{t}\|y(s)\|^2\, ds \\ &\leq c_0\|w(t_0)\|^2 + c_1(k(t) - k(t_0))\end{aligned}$$

using the dynamics of $k(\cdot)$ for the last inequality. Writing $c_2 = \|QC_{p-1}\| + \|QM^{-1}\|$, for all $x = (w, y, k) \in \mathbb{R}^{mp+1}$

$$\begin{aligned}\langle \nabla W_2(y), \eta \rangle &\leq \max_{\phi \in \mathcal{F}(w,y)}\{\langle Qy, M^{-1}\phi \rangle\} + \max_{u \in \Psi(x)}\{\langle Qy, M^{-1}Bu \rangle\} + \langle -L_3 w + C_{p-1} y \rangle \\ &\leq f(w,y)\|QM^{-1}\|\|y\| + \|QC_{p-1}\|\|y\|^2 \\ &\qquad + \|QL_3\|\|y\|\|w\| + \max_{u \in \Psi(x)}\{-\langle Qy, M^{-1}Bu \rangle\}.\end{aligned}$$

for all $\eta \in F_2(x)$. Since $M^{-1}B$ is essentially column diagonally dominant and invertible, Lemma 3.5.1 gives for $u \in \Psi(x)$

$$-\langle Qy, M^{-1}Bu \rangle \leq -k[f(w,y) + \|y\|]\nu(k)\|y\|$$

where $\nu(\cdot)$ is defined as in Lemma 3.5.2. This gives for all $x = (w, y, k) \in \mathbb{R}^{mp+1}$,

$$\langle \nabla W_2(y), \eta \rangle \leq \|QL_3\|\|y\|\|w\| + (c_2 - k\nu(k))[f(w,y)\|y\| + \|y\|^2],$$

for all $\eta \in F_2(x)$, and consequently

$$\frac{dW_2(y(t))}{dt} \leq \|QL_3\|\|y(t)\|\|w(t)\| + (c_2 - k(t)\nu(k(t)))\dot{k}(t).$$

Therefore for $t \in [t_0, \omega)$

$$0 \leq W_2(y(t)) \leq W_2(y(t_0)) + c_0\|QL_3\|\|w(t_0)\|^2 + (c_1\|QL_3\| + c_2)(k(t) - k(t_0)) - \int_{k(t_0)}^{k(t)} \theta\nu(\theta)\, d\theta. \tag{3.11}$$

Suppose for a contradiction $k(\cdot)$ is unbounded. Therefore there exists a $T \geq t_0$ such that $k(t) \geq 1$ for $t \geq T$. Setting $c^* = W_2(y(t_0)) + c_0\|QL_3\|\|w(t_0)\|^2 + (c_1\|QL_3\| + c_2)$, for $t \geq T$

$$0 \leq \liminf_{t \uparrow \omega} \frac{W_2(y(t))}{k(t)} \leq c^* - \limsup_{\zeta \to \infty} \frac{1}{\zeta} \int_{k(t_0)}^{\zeta} \theta\nu(\theta)\, d\theta.$$

By Lemma 3.5.2 this is a contradiction. Therefore $k(\cdot)$ is bounded. From (3.11) and the boundedness of $k(\cdot)$, the boundedness of $y(\cdot)$ is established. Define the function

$$W : x \mapsto W_1(w) + W_2(y).$$

For all $x = (w, y, k)$,

$$\begin{aligned}\langle \nabla W(x), \phi \rangle &\leq \langle Pw, L_1 w + L_2 y \rangle + \|QL_3\|\|y\|\|w\| + (c_2 - k\nu(k))[f(w,y)\|y\| + \|y\|^2] \\ &\leq -\frac{1}{4}(\|w\|^2 + \|y\|^2) + \frac{1}{4}(\|y\|^2 - \|w\|^2) + (\|PL_2\| + \|QL_3\|)\|w\|\|y\| \\ &\qquad + (c_2 - k\nu(k))[f(w,y)\|y\| + \|y\|^2]\end{aligned}$$

for all $\phi \in F(x)$. Note

$$-\frac{1}{4}\|w\|^2 + (\|PL_2\| + \|QL_3\|)\|w\|\|y\| = -\frac{1}{4}[(\|w\| - c_3\|y\|)^2 - c_3^2\|y\|^2]$$

where $c_3 = 2(\|PL_2\| + \|QL_3\|)$. So defining

$$k^\dagger := \frac{1}{4} + \|QM^{-1}\| + \|QC_{p-1}\| + (\|PL_2\| + \|QL_3\|)^2,$$

$$\langle \nabla W(x), \phi \rangle \leq -\frac{1}{4}(\|w\|^2 + \|y\|^2) + (k^\dagger - k\nu(k))[f(w,y)\|y\| + \|y\|^2]$$

for all $\phi \in F(x)$, and consequently, for almost all $t \in [t_0, \omega)$

$$\frac{dW(x(t))}{dt} \leq -\frac{1}{4}(\|w(t)\|^2 + \|y(t)\|^2) + (k^\dagger - k(t)\nu(k(t)))\dot{k}(t).$$

For $t \geq t_0$,

$$0 \leq W(x(t)) \leq W(x(t_0)) + \int_{t_0}^{t} (k^\dagger - k(t)\nu(k(t)))[f(w,y)\|y\| + \|y\|^2]$$

Since $k(\cdot)$ and $y(\cdot)$ are bounded, the boundedness of $w(\cdot)$ is established. Therefore $x(\cdot) = (w(\cdot), y(\cdot), k(\cdot))$ is bounded, implying $\omega = \infty$ by Theorem $A.3.3$ in the appendix. All that remains to be proved is that $(w(t), y(t)) \to (0,0)$ as $t \to \infty$. This is done by defining

$$V : x \mapsto W(x) - \int_{k(t_0)}^{k} (k^\dagger - \theta\nu(\theta))\, d\theta.$$

For all $x = (w, y, k)$,

$$\langle \nabla V(x), \phi \rangle \leq -\frac{1}{4}(\|w\|^2 + \|y\|^2).$$

for all $\phi \in F(x)$. From Theorem $A.3.7$ in the appendix, the result is established. □

## 3.6 Examples

Now we consider the control strategy of this chapter to stabilise the two examples given in the previous chapter. Note, however we now require the matrix $M^{*-1}B^*$ to be invertible and column diagonally dominant. But the matrices

$$M^{*-1}B^* = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \text{ and } M^{*-1}B^* = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix},$$

satisfy this requirement.

### 3.6.1 Example 1

Given the system of example 1 of the previous chapter, the following results, Figures $(3-1)$ and $(3-2)$, are obtained under the control strategy of (3.6).

In this example, the unmixing matrix $U \in \mathcal{I}_n^E$ is $U = I_n$, the identity matrix. In Figure $(3-1)$,

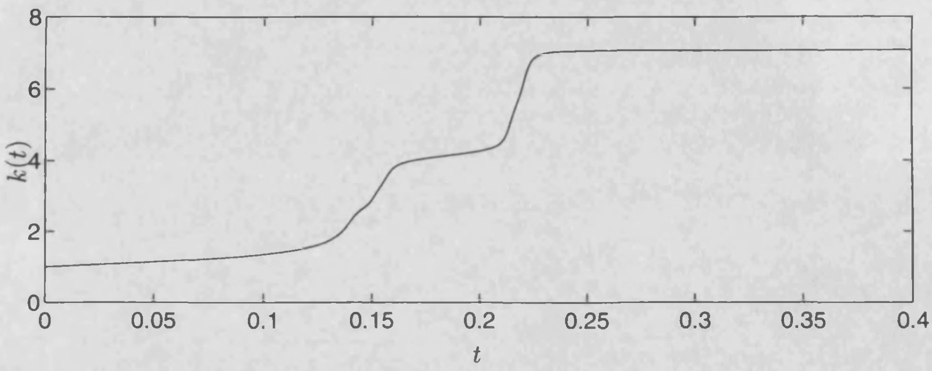


Figure 3-1: Evolution of $k$.

the adaptive gain $k(t)$ tends to a value such that $k \in T_{j-1} \cup T_j$ where $j$ is such that $K_j = I_n$ in the definition of (3.4).

Figure $(3-2)$ shows the constructed output $z(t)+\dot{z}(t)$ tends to zero as time increases as expected.

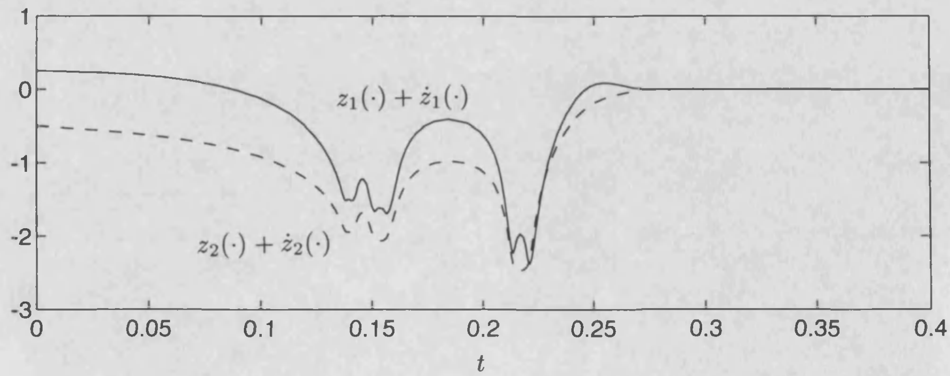


Figure 3-2: Evolution of the constructed output.

From here $z(t) \to 0$ as $t \to \infty$.

### 3.6.2 Example 2

Using example 2 of the previous chapter we obtain the results in Figures $(3-3)$ and $(3-4)$.

In Figure $(3-3)$, we again obtain similar results as those of example 1, but this time the value

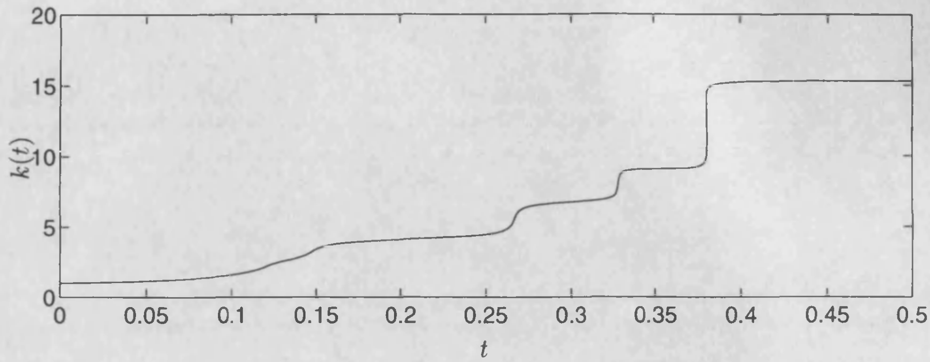


Figure 3-3: Evolution of $k$.

of $k(t)$ tends to a different limit. The limit is such that $k \in T_{j-1} \cup T_j$, where $j$ is such that

$$K_j = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

in the definition of (3.4).

Figure $(3-4)$ shows the evolution of the constructed output. Again this is similar to the

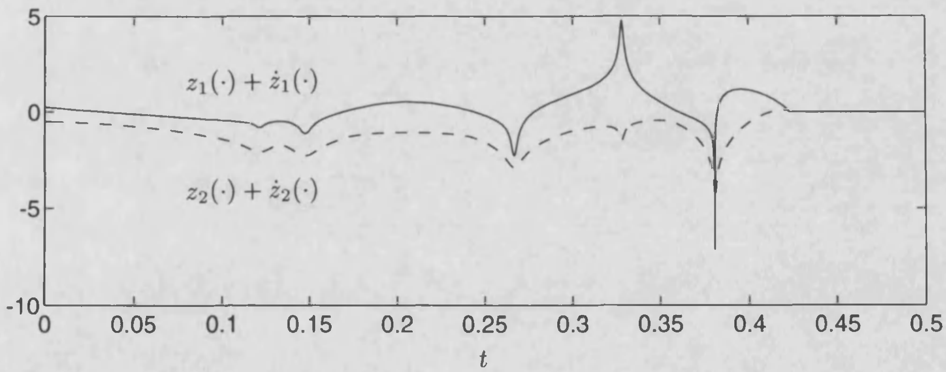


Figure 3-4: Evolution of the constructed output.

evolution of the first example, tending to zero as $t \to \infty$. From here $z(t) \to 0$ as $t \to \infty$.

## 3.7 Perturbed linear systems and output feedback

Similar to Chapter 2, the class of system studied so far in this chapter have had the full state available for feedback purposes. Here we indicate how the control strategy of system (3.6) can be carried over to the following class of nonlinearly-perturbed linear systems:

$$\left.\begin{aligned} &\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}[u(t) + g(t,\bar{x}(t))] + d(t,\bar{x}(t)),\ \bar{x}(t) \in \mathbb{R}^n,\ u(t) \in \mathbb{R}^m,\ \bar{x}(t_0) = x^0 \\ &\bar{y}(t) = \bar{C}\bar{x}(t), \qquad \bar{y}(t) \in \mathbb{R}^k. \end{aligned}\right\} \tag{3.12}$$

under the following assumptions:

**Assumptions**

**(AA1)** For some known $\bar{D} \in \mathbb{R}^{m\times k}$, the triple $(\bar{D}\bar{C}, \bar{A}, \bar{B})$ defines a minimum phase linear system of relative degree one, that is,

$$\text{rank}\begin{bmatrix} sI - \bar{A} & \bar{B} \\ \bar{D}\bar{C} & 0 \end{bmatrix} = n + m, \qquad \forall s \in \bar{\mathbb{C}}_+.$$

where $\bar{\mathbb{C}}_+$ denotes the closed right half complex plane, and

$$\hat{B} := \bar{D}\bar{C}\bar{B} \in Gl(m;\mathbb{R})$$

with $\hat{B}$ essentially column diagonally dominant.

**(AA2)** (i) For each $\bar{x} \in \mathbb{R}^n$, the function $g(\cdot,\bar{x})$ is measurable, (ii) for almost all $t \in \mathbb{R}$, $g(t,\cdot)$ is continuous and (iii) there exist a scalar $\mu_1 > 0$ and a continuous function $\gamma$ such that, for almost all $t \in \mathbb{R}$,

$$\|g(t,\bar{x})\| \leq \mu_1\gamma(\bar{C}\bar{x}), \qquad \forall \bar{x} \in \mathbb{R}^n$$

**(AA3)** (i) For all $\bar{x} \in \mathbb{R}^n$, $d(\cdot,\bar{x})$ is measurable, (ii) for almost all $t \in \mathbb{R}$, $d(t,\cdot)$ is continuous and (iii) the function $d$ is bounded in the following manner:

$$\|d(t,\bar{x})\| \leq \mu_2\|\bar{D}\bar{C}\bar{x}\| \qquad \forall (t,\bar{x}) \in \mathbb{R}\times\mathbb{R}^n$$

where $\mu_2$ can be an unknown constant.

Notice the assumption **(AA1)** requires the existence of a known matrix $\bar{D}$ so that the resulting system, with constructed output $\bar{D}\bar{C}\bar{x} \in \mathbb{R}^m$, is minimum phase.

### 3.7.1 Co-ordinate transformation

For the purposes of analysis we make the same co-ordinate transformation as made in Chapter 2. The resulting system is

$$\left.\begin{aligned} &\dot{w}(t) - L_1w(t) - L_2y(t) \in \{v \mid \|v\| \leq \mu_2\|T\|\|y\|\} \\ &\dot{y}(t) + L_3w(t) + L_4y(t) - \hat{B}u \in \mu^*\mathcal{F}(\bar{y}) \end{aligned}\right\} \tag{3.13}$$

where $\mathcal{F}(\bar{y}) = f(\bar{y})\bar{\mathbb{B}}_m$, $\mu^* = \mu_1\|\hat{B}\| + \mu_2\|\bar{D}\bar{C}\|$ and $f(\bar{y}) := \gamma(\bar{y}) + \|\bar{y}\|$. This system, in the absence of the set in the $w$ dynamics can be identified can be identified with (2.6) by setting $M = I$, $B = -\hat{B}$ and $C_{p-1} = -L_4$.

By the minimum-phase condition of **(AA1)** $\sigma(L_1) \subset \mathbb{C}_-$.

### 3.7.2 Adaptive output feedback strategy

Once again this transforms the system (3.12) to a format similar to that of the transformed system (2.6) on which the analysis of previous sections was done. This results in a similar control structure as previously encountered. Let $K(\cdot)$ and $\psi(\cdot)$ be as before. Our adaptive feedback strategy is

$$u(t) \in \bar{\Psi}(\bar{y}(t), k(t)), \qquad \dot{k}(t) = f(\bar{y}(t))\|\bar{D}\bar{y}(t)\| + \|\bar{D}\bar{y}(t)\|^2, \qquad k(0) = k^0$$

where

$$\bar{\Psi}(\bar{y}, k) := -k[f(\bar{y}) + \|\bar{D}\bar{y}\|]K(k)\psi(\bar{D}\bar{y})$$

Analogous arguments to that in previous sections of this chapter proves this strategy is a universal stabiliser for (3.12), with the same observation as made in Chapter 2.

# Chapter 4

# Adaptive stabilisation of single-input nonlinear systems

## 4.1 Introduction

In this chapter, the nonlinear systems to be stabilised will have more general nonlinearities when compared to the essentially linear systems with nonlinear perturbations encountered in the previous two chapters. Here the nonlinear systems are more general forms of the Isidori and Byrnes normal form [39], also found in their papers [9], [12], [10], [8] and [11] in the multi-input case. Initially, the special case of single-input and single-output is considered, establishing firm foundations on which to build when moving to the multi-input, multi-output case in the following chapter. The full state will not be available for feedback purposes.

Specifically, the type of system to be studied will be subclasses of the class $\mathcal{C}$ of systems that take the form,

$$\left.\begin{aligned} \dot{x}(t) &= f(t, x(t), y(t)) \\ \dot{y}(t) &= g(t, x(t), y(t)) + h(t, x(t), y(t))u \end{aligned}\right\} \tag{4.1}$$

where $y(t)$, $u(t) \in \mathbb{R}$ and $x(t) \in \mathbb{R}^n$, with $n \geq 1$. $x(t)$ is the state vector, $y(t)$ the output and $u(t)$ the input. The point $(x, y) = (0, 0)$ is an equilibrium of the system. (Note that this is a general $(n+1)$-dimensional system, but with a restriction in the dimension of the input and output.)

If the functions $f$, $g$ and $h$ do not depend explicitly on $t$, the system (4.1) is of the Isidori normal form, [39]. A transformation that transforms the system

$$\left.\begin{aligned} \dot{z}(t) &= a(z(t)) + b(z(t))u \\ y(t) &= c(z(t)) \end{aligned}\right\} \tag{4.2}$$

into the Isidori normal form, can be found in the reference [39]. Here $z(t) \in \mathbb{R}^{n+1}$ and $y(t)$, $u(t) \in \mathbb{R}$. The point $z = 0$ is an equilibrium. (A sufficient condition for this transformation to exist is that the system has relative degree 1 for all $z \in \mathbb{R}^{n+1}$. The definition of relative degree for nonlinear systems can be found in [39], but the reader can think of this concept as the following: suppose at some time $t_0$ the system is in the state $z(t_0) = z_0$. The relative degree $r$ at $z_0$ is exactly equal to the number of times the output has to be differentiated at time $t_0$ in order to have the value $u(t_0)$ of the input appearing explicitly. When (4.2) is the linear system, $a(z) = Az$, $b(z) = B$ and $c(z) = Cz$, where $(A, B, C) \in \mathbb{R}^{(n+1)\times(n+1)} \times \mathbb{R}^{(n+1)\times 1} \times \mathbb{R}^{1\times(n+1)}$, the relative degree is the integer $r$ such that $CA^k B = 0$ for all $k < r - 1$ and $CA^{r-1}B \neq 0$. This conforms with the definition that the relative degree is the *difference* between the degree of the denominator polynomial and the degree of the numerator polynomial of the transfer function $G(s) = C(sI - A)^{-1}B$ of the system, in the single-input, single-output case).

Three non-exclusive, but different subclasses of $\mathcal{C}$ shall be considered. Each subclass will have a different type of stability imposed on the zero dynamics. This being the sub-system of (4.1), namely $\dot{x}(t) = f(t, x(t), 0)$, i.e. when $y(\cdot) \equiv 0$. The idea in constructing the control is to drive the output $y(t)$ to zero, and then let the zero dynamics take the residual state to zero. We impose a condition of time invariance on the zero dynamics, ie. for all $(t, x)$, $f(t, x, 0) = \bar{f}(x)$. In two of the subclasses of $\mathcal{C}$ the zero dynamics are assumed asymptotically stable. In the third, the zero dynamics are assumed exponentially stable. This latter case will allow for uncertain functions, $g(\cdot,\cdot,\cdot)$ and the *residue* function, $f^*(t, x, y) := f(t, x, y) - \bar{f}(x)$, of greater generality. The first subclass of (4.1) to be considered has been inspired by a paper by Isidori and Byrnes, [9]. There they state that a particular member of the class $\mathcal{C}$ is *not stabilisable* by a feedback control which is a *smooth* function of the output. The reader should note that the feedbacks used throughout this thesis are essentially discontinuous in nature; they are embedded into a differential inclusion for analytical purposes. We shall find that the Isidori and Byrnes example can be *stabilised* by a *non-smooth* control of this form.

Given the stability of the zero dynamics of the system, Inverse Liapunov theory will be invoked in the analysis. By imposing certain conditions on the system, depending on the type of stability available, the existence of a Liapunov function can be guaranteed. The relevant theory can be found in Appendix B of this thesis.

The input-connection function $h(\cdot,\cdot,\cdot)$ will belong to a class of sign-definite continuous functions. In fact the function $h(\cdot,\cdot,\cdot)$ is assumed bounded away from zero while at the same time having its absolute value bounded above in a known manner. Once more the use of Nussbaum functions will be used in the feedback control to overcome the lack of knowledge of the sign of $h(\cdot,\cdot,\cdot)$. The type of Nussbaum function will be the scaling-invariant Nussbaum function as found in [46], see (1.2).

The control will again be discontinuous in nature. Once more the analytical difficulties that could arise from this, can be overcome by using differential inclusion theory. The control will be an inclusion, and this in turn will give rise to the feedback system being embedded in a differential inclusion.

## 4.2 A class $\mathcal{C}$ of systems

Three different subclasses, $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$, of the class $\mathcal{C}$ of systems of the form

$$\left.\begin{array}{ll}\dot{x}(t) = f(t,x(t),y(t)), & x(t_0) \in \mathbb{R}^n \\ \dot{y}(t) = g(t,x(t),y(t)) + h(t,x(t),y(t))u, & y(t_0) \in \mathbb{R}\end{array}\right\} \tag{4.3}$$

shall be considered. Different structural assumptions will be imposed on the functions $f : \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}^n$ and $g : \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}$, dependent on the subclass of system. All three subclasses of $\mathcal{C}$ will have some common structural assumptions. These are presented here, and shall hereafter be refered to as the general structural assumptions of the class $\mathcal{C}$.

**General Structural Assumptions:**

**(G1):** All functions $f(\cdot,\cdot,\cdot)$, $g(\cdot,\cdot,\cdot)$ and $h(\cdot,\cdot,\cdot)$ are continuous.

**(G2):** The function $h : \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}$ is bounded away from zero, and its absolute value is bounded above in the following way. There exists an $\epsilon > 0$, a known continuous function $\gamma : \mathbb{R}\times\mathbb{R}\to\mathbb{R}_+$ with $\gamma(t,y) \geq \epsilon$ for all $(t,y) \in \mathbb{R}\times\mathbb{R}$, and scalars $\alpha$, $\beta \in \mathbb{R}_+$ (not necessarily known) with $0 < \alpha \leq \beta$ such that

(i) $0 < \alpha\gamma(t,y) \leq |h(t,x,y)| \leq \beta\gamma(t,y)$, for all $(t,x,y) \in \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}$.

(ii) $K \subset \mathbb{R}$ bounded $\implies \gamma(\mathbb{R},K)$ bounded.

**(G3):** $(x,y) = (0,0)$ is the unique stationary point of the system (4.3) in the absence of control.

**(G4):** The zero dynamics function, $f(\cdot,\cdot,0)$, is time invariant, ie.

$$f(t,x,0) = \bar{f}(x), \qquad \forall (t,x) \in \mathbb{R}\times\mathbb{R}^n.$$

for some continuous function $\bar{f}(\cdot)$.

These four general structural assumptions will be common to all subclasses of the class $\mathcal{C}$. Structural assumption **(G2)** provides information on the growth rate of the input connection function, making sure the control $u$ has an effect on the system in a known manner. **(G1)** is made for analytical purposes, so the existence of a solution to a differential inclusion in which (4.3) will be embedded, will be guaranteed. Structural assumption **(G3)** makes the origin, $(x, y) = (0, 0)$, the target for the control.

If the initial conditions and input $u(\cdot)$ in (4.3) are such that $y(\cdot) = 0$, then the residual dynamics are called the zero dynamics, ie. the dynamics

$$\dot{x}(t) = \bar{f}(x(t)), \qquad x(0) \in \mathbb{R}^n.$$

**(G4)** is imposed on the zero dynamics in order that the inverse Liapunov functions, for this stable system, are time invariant. Each subclass of $\mathcal{C}$ will have a different form of stability imposed on the zero dynamics. The idea being to drive the output $y(t)$ to zero under the effects of the control, and then let the state component $x(\cdot)$ be driven to the target under the effects of the zero dynamics of the system.

This idea leads to the definition of the residue function $f^*(\cdot,\cdot,\cdot)$ of $f(\cdot,\cdot,\cdot)$, which is defined in the following way:

$$f^*(t, x, y) := f(t, x, y) - f(t, x, 0). \tag{4.4}$$

## 4.3 A universal control

The control given here will achieve the objective of making the origin $(x, y) = (0, 0)$ globally attractive for all subclasses of $\mathcal{C}$. The control is discontinuous in nature, formally being given as:

$$\left.\begin{aligned} u(t) &= \nu(k(t))\rho(y(t))|y(t)|^{-1}y(t) \\ \dot{k}(t) &= |y(t)|\rho(y(t))\gamma(t, y(t)). \end{aligned}\right\} \tag{4.5}$$

The function $\nu(\cdot)$ is a continuous, scaling invariant Nussbaum function, [46], [45] and refer to (1.2). The function $\rho(\cdot)$ is a known continuous function which will be dependent on the subclass of $\mathcal{C}$. The function $\gamma(\cdot,\cdot)$ is the function found in general assumption **(G2)**. As mentioned, the control has a discontinuity at $y = 0$. The analytical problems that could arise from this discontinuity can be overcome, as in the previous chapters, by embedding the control in a differential inclusion. Therefore the discontinuous control (4.5) can be interpreted in the set-valued sense

$$\left.\begin{aligned} &u(t) \in \Psi(k(t), y(t)) := \nu(k(t))\rho(y(t))\psi(y(t)) \\ &\dot{k}(t) = |y(t)|\rho(y(t))\gamma(t, y(t)) \end{aligned}\right\} \tag{4.6}$$

where $\psi(\cdot)$ is a set-valued map, defined as in (2.7). $\psi(\cdot)$ is an upper-semicontinuous set-valued map, which in turn makes $\Psi(\cdot,\cdot)$ an upper-semicontinuous map, by Lemma $A$.2.6 in the appendix. $\Psi(\cdot,\cdot)$ has convex and compact values, since $\psi(\cdot)$ has these properties.

The particular choice of $\rho(\cdot)$, dependent on the subclass of $\mathcal{C}$, will be discussed in the relevant section where the subclass is considered.

Therefore the class $\mathcal{C}$, under this feedback control, can be embedded into the following differential inclusion. Let $w(\cdot) := (x(\cdot), y(\cdot), k(\cdot))$,

$$\dot{w}(t) \in F(t, w(t)), \qquad w(t_0) = (x(t_0), y(t_0), k(t_0)) \tag{4.7}$$

where $F(t, w) := F_1(t, w) \times F_2(t, w) \times F_3(t, w)$ with

$$\begin{aligned} F_1(t, w) &:= \{f(t, x, y)\} \\ F_2(t, w) &:= \{g(t, x, y) + h(t, x, y)u \mid u \in \Psi(k, y)\} \\ F_3(t, w) &:= \{|y|\rho(y)\gamma(t, y)\} \end{aligned}$$

Note that $F(\cdot,\cdot)$ is an upper-semicontinuous set-valued map with non-empty, convex and compact values. A solution (not necessarily unique) exists on a maximal interval of existence by Theorem $A$.3.1 and Theorem $A$.3.2. Call this solution $w : [t_0, \omega) \to \mathbb{R}^{n+2}$

## 4.4 A subclass inspired by Byrnes and Isidori

The first of the subclasses of the class $\mathcal{C}$ will be a subclass inspired by the example given in Byrnes and Isidori, [9]. They consider the autonomous system

$$\left.\begin{aligned} &\dot{x}(t) = y^2(t) - x^5(t) \\ &\dot{y}(t) = x^2(t) + u, \end{aligned}\right\} \tag{4.8}$$

and prove that no control of the form

$$u = \xi(y),$$

where $\xi(\cdot)$ is a smooth function, will stabilise the system. Clearly the non-smooth controls used in this chapter do not fall into this category. In fact in this section, we shall show that the universal control (4.6) not only stabilises the system (4.8) but all systems of class $\mathcal{C}$ to which (4.8) belongs.

The subclass $C_1$ of $C$ to be considered in this section is classified by the structural assumptions made on the system (4.3). These assumptions are made with the Byrnes and Isidori example (4.8), in mind. The structural assumptions are:

**Structural Assumptions:**

**(A1.1):** The general assumptions, **(G1)**, **(G2)**, **(G3)** and **(G4)** of section **(4.2)** hold.

**(A1.2):** The following form of stability is imposed on the zero dynamics of the system (4.3), $\dot{x}(t) = f(t, x(t), 0) = \bar{f}(x(t))$: there exists a possibly unknown $C^1$ function $W : \mathbb{R}^n \to \overline{\mathbb{R}}_+$ with the properties:

(i) $W(0) = 0$,

(ii) $W(x) > 0$ for $x \neq 0$ and $W(x) \to \infty$ as $\|x\| \to \infty$,

(iii) $\langle \nabla W(x), \bar{f}(x) \rangle \leq -\gamma_0 \|x\|^{a_1}$, for all $x \in \mathbb{R}^n$, where $\gamma_0 > 0$ and $a_1 > 1$ are possibly unknown,

(iv) $\|\nabla W(x)\| < \gamma_1 \|x\|^{a_1 - a_2}$, for all $x \in \mathbb{R}^n$, where $0 < a_2 < a_1$ and $\gamma_1 > 0$ are possibly unknown.

For some known continuous function $\rho : \mathbb{R} \to \overline{\mathbb{R}}_+$ satisfying, for all $r \in [0, \infty)$, $|y|^r \leq \gamma_{6,r}\rho(y)$ for all $y \in \mathbb{R}$, where $\gamma_{6,r}$ is a (possibly unknown) constant that can depend on $r$, the following hold.

**(A1.3):** The residue function belongs to the class of functions with the following bound:

$$\|f^*(t, x, y)\| \leq \gamma_2 |y| h^*(y)(1 + \|x\|^{\delta_1}), \qquad \forall (t, x, y) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$$

for some possibly unknown $\gamma_2 > 0$, and $0 \leq \delta_1 < a_2$ with $a_2$ as in **(A1.2)**. $h^* : \mathbb{R} \to \overline{\mathbb{R}}_+$ is a function that satisfies for all $r$ such that $1 \leq r < \infty$

$$(h^*(y))^r \leq \gamma_{3,r}\rho(y), \qquad \forall y \in \mathbb{R}$$

where $\gamma_{3,r}$ is a constant that can depend on $r$.

**(A1.4):** The function $g(\cdot, \cdot, \cdot)$ is a continuous function belonging to the class of functions with the following bound:

$$|g(t, x, y)| \leq \gamma_4 l(y)(1 + \|x\|^{\delta_2}), \qquad \forall (t, x, y) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}.$$

for $\gamma_4 > 0$ and $0 \leq \delta_2 < a_1$, possibly unknown, with $a_1$ as in **(A1.2)**. $l : \mathbb{R} \to \mathbb{R}_+$ is a function that satisfies for all $r$ such that $1 \leq r < \infty$,

$$(l(y))^r \leq \gamma_{5,r}\rho(y), \qquad \forall y \in \mathbb{R}$$

where $\gamma_{5,r}$ is a constant that can depend on $r$.

Note that the bound in **(A1.3)** has a factor of $|y|$. Also the bound with respect to $x$ is stronger in **(A1.3)** than in **(A1.4)**.

The reader can think of $\rho(y)$ as the function $\exp(|y|)$, and $h^*(y)$ and $l(y)$ as functions bounding polynomial growth, ie. $h^*(y) = l(y) = 1 + |y|^\lambda$ where $\lambda \in \mathbb{R}_+$.

The analysis can now be given that establishes (i) the maximal interval of existence is the infinite interval $[t_0, \infty)$, (ii) the adaptive gain $k(\cdot)$ tends to a finite limit and (iii) the origin is globally attractive with respect to the solutions $x(t)$ and $y(t)$. This result is given in the following theorem:

**Theorem 4.4.1** *Let $w : [t_0, \omega) \to \mathbb{R}^{n+2}$ be a solution to the differential inclusion (4.7), on its maximal interval of existence, then:*

*(i) $\omega = \infty$,*

*(ii) $\lim_{t\to\infty} k(t)$ exists and is finite,*

*(iii) $(x(t), y(t)) \to (0,0)$ as $t \to \infty$.*

**Proof:** Recall Young's inequality: for $a$, $b \in \mathbb{R}$,

$$|ab| \leq \frac{1}{p}|a|^p + \frac{1}{q}|b|^q$$

where $\frac{1}{p} + \frac{1}{q} = 1$, see [24]. Given any constants $c_1 > 0$, $c_2 \geq 0$, $c_3 \geq 0$ and $0 < c_4 < a_1$, there exist $b_1 > 0$ and $b_2 > c_3$, such that

$$c_2|y|^{c_3}\|x\|^{c_4} \leq b_1|y|^{b_2} + c_1\|x\|^{a_1} \tag{4.9}$$

by equating $p = a_1c_4^{-1}$, $q = a_1(a_1 - c_4)^{-1}$, $b_1 = q^{-1}[c_2(c_1p)^{(-p^{-1})}]^q$ and $b_2 = qc_3$.

Let $V(x,y) := W(x) + \frac{1}{2}y^2$. For all $(t, w) \in \mathbb{R} \times \mathbb{R}^{n+2}$,

$$\begin{aligned}
\langle \nabla V(x,y), \eta\rangle &\leq \langle \nabla W(x), \bar{f}(x) + f^*(t,x,y)\rangle + yg(t,x,y) + \max_{u\in\Psi(k,y)}\{yh(t,x,y)u\} \\
&\leq -\gamma_0\|x\|^{a_1} + \gamma_1\gamma_2\|x\|^{a_1-a_2}|y|h^*(y)(1 + \|x\|^{\delta_1}) \\
&\qquad + \gamma_4|y|l(y)(1 + \|x\|^{\delta_2}) + \nu(k)h(t,x,y)|y|\rho(y) \\
&\leq -\frac{1}{2}\gamma_0\|x\|^{a_1} + c^\dagger|y|\rho(y) + \nu(k)h(t,x,y)|y|\rho(y)
\end{aligned}$$

for all $\eta \in F_1(t,w) \times F_2(t,w)$, for some constant $c^\dagger$, by use of (4.9) and **(A1.2)** to **(A1.4)**. (Note in considering terms $|y|^{r-1}h^{*r}(y)$ for $r \geq 1$, we have used the fact that if $|y|^{2(r-1)} \leq \gamma_{6,2(r-1)}\rho(y)$

for some $1 \le r < \infty$ then $|y|^{(r-1)} \le \sqrt{\gamma_{6,2(r-1)}}\sqrt{\rho(y)}$.) So

$$\langle \nabla V(x,y), \eta \rangle \le -\frac{1}{2}\gamma_0 \left(\|x\|^{a_1} + |y|^2\right) + \epsilon^{-1}(c^\dagger + \frac{1}{2}\gamma_0\gamma_{6,1})|y|\rho(y)\gamma(t,y) + \mathcal{N}(k)\gamma(t,y)|y|\rho(y) \tag{4.10}$$

where

$$\mathcal{N}(k) := \begin{cases} \gamma\nu(k), & \nu(k) \ge 0 \\ \delta\nu(k), & \nu(k) < 0 \end{cases} \tag{4.11}$$

with constants $\gamma$, $\delta > 0$. (Note that the sign of $h(\cdot,\cdot,\cdot)$ never changes, and so $\gamma$ and $\delta$ will be fixed. In fact $\gamma = \beta$ when $h(t,x,y) > 0$ or $\gamma = \alpha$ when $h(t,x,y) < 0$, and $\delta = \alpha$ when $h(t,x,y) > 0$ or $\delta = \beta$ when $h(t,x,y) < 0$.) Since $\nu(\cdot)$ is a continuous, scaling invariant Nussbaum function, $\mathcal{N}(\cdot)$ is a continuous Nussbaum function.

Therefore

$$\begin{aligned} \frac{dV(x(t),y(t))}{dt} \le -\frac{1}{2}\gamma_0 \left(\|x(t)\|^{a_1} + |y(t)|^2\right) + \epsilon^{-1}(c^\dagger + \frac{1}{2}\gamma_0\gamma_{6,1})|y(t)|\rho(y(t))\gamma(t,y(t)) \\ + \mathcal{N}(k(t))\gamma(t,y(t))|y(t)|\rho(y(t)). \end{aligned} \tag{4.12}$$

Recall that $\dot{k}(t) = |y(t)|\gamma(t,y(t))\rho(y(t))$, and with (4.12) this gives

$$\begin{aligned} 0 \le W(x(t)) + \frac{1}{2}y^2(t) \le W(x(t_0)) + \frac{1}{2}y^2(t_0) \\ + \epsilon^{-1}(c^\dagger + \frac{1}{2}\gamma_0\gamma_{6,1})(k(t) - k(t_0)) + \int_{k(t_0)}^{k(t)} \mathcal{N}(k)\, dk. \end{aligned} \tag{4.13}$$

Suppose for a contradiction $k(\cdot)$ is unbounded. Since $\dot{k}(t) \ge 0$, we can assume the existence of a $\tau > t_0$ such that for all $t > \tau$, $k(t) \ge 1$. This and (4.13) gives

$$0 \le c^* + \liminf_{k\to\infty} \frac{1}{k}\int_{k(t_0)}^{k} \mathcal{N}(k)\, dk$$

where $c^* := \epsilon^{-1}(c^\dagger + \frac{1}{2}\gamma_0\gamma_{6,1}) + (2W(x(t_0)) + y^2(t_0))/2$. This is clearly a contradiction since $\mathcal{N}(\cdot)$ is a Nussbaum function. Hence the boundedness of $k(\cdot)$.
From the boundedness of $k(\cdot)$, the continuity of $\mathcal{N}(\cdot)$ and (4.13), the boundedness of $x(\cdot)$ and $y(\cdot)$ are established. This in turn, by Theorem $A.3.3$ in the appendix, gives $\omega = \infty$.

It remains to prove $(x(t), y(t)) \to (0,0)$ as $t \to \infty$. This is done by defining

$$V^*(x,y,k) := V(x,y) - \left(\epsilon^{-1}(c^\dagger + \frac{1}{2}\gamma_0\gamma_{5,2})k + \int_{k(t_0)}^{k} \mathcal{N}(k)\, dk\right).$$

Noting that

$$\langle \nabla V^*(x, y, k), \phi \rangle \le -\frac{1}{2}\gamma_0(\|x\|^{a_1} + y^2)$$

for all $\phi \in F(t, w)$, so by Theorem $A.3.7$ in the appendix, $(x(t), y(t)) \to (0,0)$ as $t \to \infty$. □

### 4.4.1 Positively-homogeneous functions

We now give a class of functions that satisfy the structural assumptions **(A1.2)** and **(A1.3)**. This is a class of functions $f(\cdot,\cdot,\cdot)$ that are positively homogeneous of order $k > 1$, with respect to the second and third arguments, ie. functions with the property

$$f(t, cx, cy) = c^k f(t, x, y), \qquad k > 1, \tag{4.14}$$

for all $c \in \bar{\mathbb{R}}_+$. With this condition imposed on $f(\cdot,\cdot,\cdot)$, the function $\bar{f}(\cdot)$ is also positively-homogeneous of order $k$, since

$$\bar{f}(cx) = f(t, cx, 0) = c^k f(t, x, 0) = c^k \bar{f}(x).$$

Similarly the function $f^*(\cdot,\cdot,\cdot)$ is positively-homogeneous of order $k$ in its second two arguments, since

$$\begin{aligned} f^*(t, cx, cy) &= f(t, cx, cy) - \bar{f}(cx) \\ &= c^k (f(t, x, y) - \bar{f}(x)) = c^k f^*(t, x, y). \end{aligned}$$

Since $\bar{f}(\cdot)$ is a positively-homogeneous function of order $k > 1$, if we impose the conditions $\bar{f} \in C^1$ and the origin of the system $\dot{x}(t) = \bar{f}(x(t))$ is globally (uniformly) asymptotically stable, by Theorem $B.2.5$ in the appendix, there exists $W : \mathbb{R}^n \to \mathbb{R}_+$ a $C^1$ function such that

$$c_1 \|x\|^{(k-1)(m-1)} \le W(x) \le c_2 \|x\|^{(k-1)(m-1)}, \qquad \forall x \in \mathbb{R}^n$$

for some constants $c_1$, $c_2 > 0$ and some $m$ sufficiently large. Also

$$\langle \nabla W(x), \bar{f}(x) \rangle = -\|x\|^{m(k-1)}, \qquad \forall x \in \mathbb{R}^n$$

and

$$\|\nabla W(x)\| < c_3 \|x\|^{m(k-1)-k}, \qquad \forall x \in \mathbb{R}^n.$$

for some constant $c_3 > 0$. So **(A1.2)** holds with this $W(\cdot)$, where $a_1 = m(k-1)$ and $a_2 = k$. The inequalities $a_1$ and $a_2$ must satisfy are true, since $m$ is any sufficiently large number.

Now we must ensure that structural assumption **(A1.3)** holds. In order for this to be true, we impose the extra conditions of $f^*(\cdot,\cdot,\cdot)$ being continuously differentiable and $f^*(\mathbb{R}, \mathbb{S}^{n+m-1})$

being bounded, where $\mathbb{S}^i$ is the unit sphere in $\mathbb{R}^{i+1}$.

Since $f^*(\cdot,\cdot,\cdot)$ is positively-homogeneous of order $k$ in its second and third argument, the function

$$\hat{f}(t,x,y) := \begin{cases} f^*(t,x,y)/\|(x,y)\|, & (x,y) \neq 0 \\ 0, & (x,y) = 0 \end{cases}$$

is a well-defined function which is homogeneous of order $k-1$ in its second and third arguments. The latter property is true, since

$$\frac{f^*(t,cx,cy)}{\|(cx,cy)\|} = c^{k-1}\frac{f^*(t,x,y)}{\|(x,y)\|}, \qquad \forall c > 0.$$

The case $c = 0$ is clear. Since we are using the two norm, $\|(x,y)\| = \left(\|x\|^2 + \|y\|^2\right)^{\frac{1}{2}}$. Therefore, by the extra condition imposed on $f^*(\cdot,\cdot,\cdot)$,

$$\begin{aligned} \|\hat{f}(t,x,y)\| &\leq K_1 \left(\|x\|^2 + \|y\|^2\right)^{\frac{k-1}{2}} \\ &\leq K_2 \left(\|x\|^{k-1} + \|y\|^{k-1}\right) \\ &\leq K_2 \left(1 + \|y\|^{k-1}\right)\left(1 + \|x\|^{k-1}\right). \end{aligned}$$

for all $(t,x,y) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$. The first inequality is true by the following argument. If $q(t,x,y)$ is homogeneous of order $k$ in its second and third arguments and $q(\mathbb{R}, \mathbb{S}^{n+m-1})$ is bounded with $q(t,0,0) = 0$ for all $t \in \mathbb{R}$, then

$$\|q(t,x,y)\| \leq K_1 \|(x,y)\|^k, \qquad \forall (t,x,y) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$$

for some constant $K_1 \geq 0$, by arguments used in Lemma $B$.2.1. The second inequality is true by the following argument: the function $(x,y) \mapsto \|x\|^k + \|y\|^k$ is positively-homogeneous of order $k$. Therefore

$$\|x\|^k + \|y\|^k \geq K^* \|(x,y)\|^k, \qquad \forall (x,y) \in \mathbb{R}^n \times \mathbb{R}^m$$

for some $K^* > 0$, by Lemma $B$.2.1. Therefore **(A1.2)** and **(A1.3)** automatically hold.

### 4.4.2 Example

Here we show the example which has inspired this section, (4.8), is indeed stabilised by the universal feedback control (4.6). All the controller need know is that the structural assumptions are satisfied with $\rho(y) = \exp(|y|)$ and $\gamma(t,y) = 1$. Note $x \mapsto -x^5$ is positively-homogeneous of order 5. So by the previous section **(A1.2)** is satisfied. $h^*(y) = |y|$, $\delta_1 = 0$, $\delta_2 = 2$ and $l(y) = 1$ for all $y \in \mathbb{R}$ satisfy the conditions. With initial conditions $x(0) = 2$, $y(0) = \frac{1}{2}$ and $k(0) = \frac{1}{2}$, the following results can be obtained.

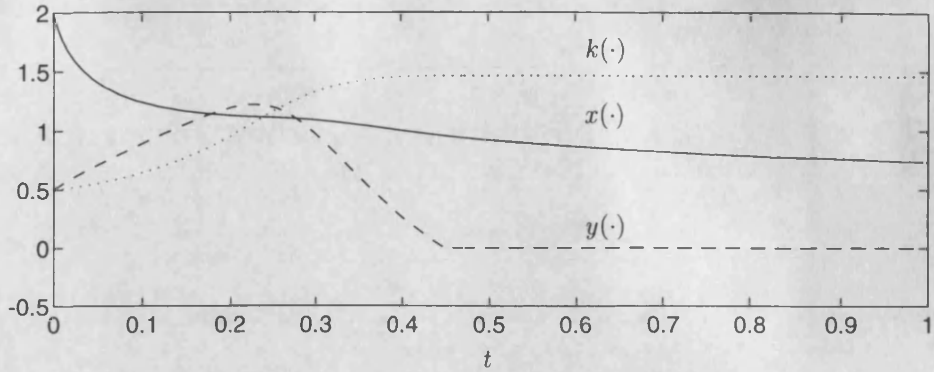


Figure 4-1: The evolution of $x(\cdot)$, $y(\cdot)$ and $k(\cdot)$.

In Figure $(4-1)$, we notice that $y$ goes to zero, and $k$ converges. $x$ does tend to zero from this point on, but slowly, as it is governed by the zero dynamics $\dot{x}(t) = -x^5(t)$. Isidori and Byrnes have shown in [9] that the above example is impossible to stabilise with a feedback of the form $u = g(y)$, where $g$ is *smooth.* Note that the universal control (4.6) is not of this form, being essentially discontinuous in nature.

## 4.5 A class with a bounded input/bounded output subsystem

The second subclass $\mathcal{C}_2$ of the class $\mathcal{C}$ to be studied has the key feature of the sub-state $x$ dynamics having the bounded input, bounded output property, see **(A2.2)** below. Once again we shall find the universal control (4.6) makes the origin globally attractive. The structural assumptions for this subclass are the following:

**Structural Assumptions:**

**(A2.1):** The general assumptions **(G1)** to **(G4)** hold.

**(A2.2):** Bounded input (i.e. $\|y\|_\infty < \infty$) for the system $\dot{x}(t) = f(t, x(t), y(t))$ implies bounded output (i.e. $\|x\|_\infty < \infty$).

**(A2.3):** For some known continuous function $\rho : \mathbb{R} \to \bar{\mathbb{R}}_+$ satisfying $\rho(y) \geq |y|^p$ for all $y \in \mathbb{R}$ for some $0 \leq p < \infty$, the function $g(\cdot, \cdot, \cdot)$ is in the class of continuous functions with the following bound:

$$|g(t, x, y)| \leq \gamma_1 \rho(y), \qquad \forall (t, x, y) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R},$$

where $\gamma_1 > 0$ is a possibly unknown constant.

**(A2.4):** The function $x \mapsto f(t, x, 0) = \bar{f}(x)$ has the following properties:

- $x = 0 \iff \bar{f}(x) = 0$.
- The origin is globally asymptotically stable for the system $\dot{x}(t) = \bar{f}(x(t))$.
- $x \mapsto \bar{f}(x)$ is locally Lipschitz.

**(A2.5):** There exists a continuous function $f^\dagger : \mathbb{R}^n \times \mathbb{R} \to \overline{\mathbb{R}}_+$ with $f^\dagger(x, 0) = 0$ for all $x \in \mathbb{R}^n$ such that

$$\|f^*(t,x,y)\| \leq f^\dagger(x,y), \qquad \forall (t,x,y) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}.$$

In structural assumption **(A2.3)** we find the most restrictive assumption that will be imposed on the function $g(\cdot,\cdot,\cdot)$ in all of the subclasses of this chapter. Here the bound is independent of $x$ and $t$.

As with the first subclass of $\mathcal{C}$, we will show that under the universal adaptive control (4.6), the origin is globally attractive.

Now we give a proposition that will be useful in the analysis of the main result of this section.

**Proposition 4.5.1** *Consider*

$$\dot{x}(t) = f(t, x(t), y(t)) = \bar{f}(x(t)) + f^*(t, x(t), y(t)), \qquad x(t_0) = x^0 \tag{4.15}$$

*with $y(t) \in \mathbb{R}^m$, $m \geq 1$, $x(t) \in \mathbb{R}^n$, and assume the following hold:*

***(P1):*** *The structural assumption **(A2.4)** holds for the function $x \mapsto \bar{f}(x)$,*

***(P2):*** *The structural assumption **(A2.5)** holds,*

***(P3):*** *The input $y(\cdot)$ is continuous and satisfies $y(t) \to 0$ as $t \to \infty$.*

*If $x(\cdot)$ is bounded, then $x(t) \to 0$ as $t \to \infty$.*

**Proof:** Property (P1) gives the existence of a function $V : \mathbb{R}^n \to \mathbb{R}$, such that

1. $V(\cdot)$ is smooth and defined on $\mathbb{R}^n$,
2. there exist continuous functions $U_1(\cdot)$, $U_2(\cdot)$ and $U_3(\cdot)$ defined on $\mathbb{R}^n$ with $U_i(0) = 0$, $U_i(x) > 0$ for $x \neq 0$, $i \in \{1, 2, 3\}$, and $U_1(x) \to \infty$ as $\|x\| \to \infty$,
3. $U_1(x) \leq V(x) \leq U_2(x)$, for all $x \in \mathbb{R}^n$,
4. $W(x) := \langle \nabla V(x), \bar{f}(x) \rangle \leq -U_3(x)$, for all $x \in \mathbb{R}^n$,

by Theorem $B$.3.1.

Since $x(\cdot)$ is bounded, there exists an $0 < R < \infty$ such that $x(t) \in R\mathbb{B}$ for all $t \in [t_0, \infty)$. Since $V(\cdot)$ is smooth there exists a $K > 0$ such that $\|\nabla V(x)\| \leq K$ for all $x \in R\mathbb{B}$.

Given $0 < \epsilon < R$, let $0 \neq c \in \text{Image}(V)$ be such that, with $V^* := \cup_{0 \leq r \leq c} V^{-1}(r)$, $V^* \subset \epsilon\mathbb{B}$. (This is possible by 2 and 3). Let $\hat{\epsilon} > 0$ be such that $\hat{\epsilon}\mathbb{B} \subset V^*$. Define $U^* : \bar{\mathbb{R}}_+ \to \bar{\mathbb{R}}_+$ as

$$U^*(r) := \min\{U_3(x) \,|\, \|x\| = r\}$$

Clearly $U^*(r) = 0 \iff r = 0$, $U^*(\|x\|) \leq U_3(x)$ and $U^*(\cdot)$ is continuous. (The latter property comes from the continuity of $U_3(\cdot)$.) Define $m := \min\{U^*(r) \,|\, r \in [\hat{\epsilon}, R]\} > 0$. Note $f^\dagger(x(t), y(t)) \to 0$ as $t \to \infty$. (For if not, for some $\delta > 0$, there exists a sequence $(t_n)$ with $t_n \to \infty$, with $f^\dagger(x(t_n), y(t_n)) \geq \delta$. For a subsequence $(t_{n_k})$, $x(t_{n_k})$ tends to a limit, $\bar{x}$ say, since $x(\cdot)$ is bounded. Therefore $f^\dagger(x(t_{n_k}), y(t_{n_k})) \to f^\dagger(\bar{x}, 0) = 0$. This would be a contradiction.) Therefore, by (a) the boundedness of $x(\cdot)$, (b) the continuity of $f^\dagger(\cdot,\cdot)$, (c) $f^\dagger(\cdot, 0) \equiv 0$ and (d) $y(t) \to 0$ as $t \to \infty$, we conclude there exists $T \geq t_0$ such that

$$\|f^*(t, x(t), y(t))\| \leq f^\dagger(x(t), y(t)) < \frac{m}{2K}$$

for all $t > T$. For such $t$,

$$\begin{aligned}
\langle \nabla V(x), f(t,x,y) \rangle &= \langle \nabla V(x), \bar{f}(x) \rangle + \langle \nabla V(x), f^*(t,x,y) \rangle \\
&< -U_3(x) + \frac{m}{2} \\
&\leq -U^*(\|x\|) + \frac{m}{2}
\end{aligned}$$

Therefore, outside $V^*$, $\langle \nabla V(x), f(t,x,y) \rangle \leq -m/2$. Therefore by Theorem A.3.8, $x(t) \to \epsilon\mathbb{B}$ as $t \to \infty$. Since $\epsilon > 0$ was arbitrary, $x(t) \to 0$ as $t \to \infty$ which proves the result. □

This proposition is used in the main theorem of this section. It establishes (i) the maximal interval of existence is the infinite interval $[t_0, \infty)$, (ii) the adaptive gain $k(\cdot)$ tends to a finite limit and (iii) the origin is globally attractive with respect to the solutions $x(\cdot)$ and $y(\cdot)$.

**Theorem 4.5.2** *Let $w : [t_0, \omega) \to \mathbb{R}^{n+2}$ be a solution to the differential inclusion (4.7) on its maximal interval of existence, then*

*(i)* $\omega = \infty$,

*(ii)* $\lim_{t\to\infty} k(t)$ *exists and is finite,*

*(iii)* $(x(t), y(t)) \to (0,0)$ *as* $t \to \infty$.

**Proof:** Firstly we shall show $k(\cdot)$ is bounded, which will also give the boundedness of $y(\cdot)$. Consider the function $Y : y \mapsto \frac{1}{2}y^2$. For all $(t, w) \in \mathbb{R} \times \mathbb{R}^{n+2}$,

$$\begin{aligned}
y\eta &\leq yg(t,x,y) + \max\nolimits_{u \in \Psi(k,y)}\{yh(t,x,y)u\} \leq \gamma_1|y|\rho(y) + \nu(k)h(t,x,y)|y|\rho(y) \\
&\leq \epsilon^{-1}\gamma_1|y|\rho(y)\gamma(t,y) + \mathcal{N}(k)|y|\rho(y)\gamma(t,y)
\end{aligned}$$

for all $\eta \in F_2(t,w)$, with $\mathcal{N}$ defined as in (4.11). Therefore

$$\frac{d\|y(t)\|^2}{dt} \leq \epsilon^{-1}\gamma_1|y(t)|\rho(y(t))\gamma(t,y(t)) + \mathcal{N}(k(t))|y(t)|\rho(y(t))\gamma(t,y(t)).$$

Recalling $\dot{k}(t) = |y(t)|\rho(y(t))\gamma(t,y(t))$, we get

$$0 \leq \frac{1}{2}y^2(t) \leq \frac{1}{2}y^2(t_0) + \epsilon^{-1}\gamma_1(k(t) - k(t_0)) + \int_{k(t_0)}^{k(t)} \mathcal{N}(k)\,dk. \tag{4.16}$$

By an analogous argument as found in Theorem 4.4.1, by assuming $k(\cdot)$ is unbounded, a contradiction arises. Therefore $k(\cdot)$ is bounded.

From $k(\cdot)$ being bounded, $\mathcal{N}(\cdot)$ being continuous and (4.16) the boundedness of $y(\cdot)$ is established. From assumption **(A2.2)** we get the boundedness of $x(\cdot)$. This in turn gives $\omega = \infty$ by Theorem $A.3.3$ in the appendix.

Now we show $y(t) \to 0$ as $t \to \infty$. This is done by showing $y(\cdot) \in L^q$, for some $1 \leq q < \infty$, $\dot{y}(\cdot) \in L^\infty$ and then using Lemma $D.2.1$ in the appendix. With $q = p + 1$, since $\dot{k}(t) = |y(t)|\rho(y(t))\gamma(t,y(t)) \geq \epsilon|y(t)|^q$, for some $q$ with $1 \leq q < \infty$, by the property imposed on $\rho(\cdot)$, and since $k(\cdot)$ is bounded, then $y(\cdot) \in L^q$. The boundedness of $\dot{y}(\cdot)$ is obtained directly from the dynamics of $y$ in (4.7), since $g(\cdot,\cdot,\cdot)$ is bounded by the continuous function $\rho(\cdot)$ and $h(\cdot,\cdot,\cdot)$ bounded by the function $\gamma(\cdot,\cdot)$ with the property $\gamma(\mathbb{R},K)$ bounded for bounded $K \subset \mathbb{R}$, the control is upper-semicontinuous and the arguments of the functions are bounded. So indeed $y(t) \to 0$ as $t \to \infty$.

All that remains to be proved is that $x(t) \to 0$ as $t \to \infty$. This is true by Proposition 4.5.1. □

### 4.5.1 Example

With $n = 2$ and $x(\cdot) = [x_1(\cdot)\ x_2(\cdot)]^T$, consider the system

$$\begin{aligned}\dot{x}(t) &= -x(t)\|x(t)\| + (\exp(y^3(t)) - 1)x(t)\\ \dot{y}(t) &= \exp(y(t))\sin(x_1(t)) + u\end{aligned}$$

This example satisfies all the necessary assumptions of this section, and is therefore stabilisable by the universal control (4.6) with $\rho(y) = \exp(y^2(t))$ and $\gamma(t,y) = 1$ for all $(t,y) \in \mathbb{R} \times \mathbb{R}$.

Letting $x_1(0) = 1$, $x_2(0) = -1$, $y(0) = 0.25$ and $k(0) = 0.5$ the following results can be obtained.

In Figure $(4-2)$, we see the control takes $y$ to zero, and then the zero dynamics take over and take $x$ to zero, but slowly for this example (being governed by $\dot{x}(t) = -x(t)\|x(t)\|$).

Note in this example the $\exp(y^3(t))$ term in the dynamics of $x$ grows faster than $\exp(y^2(t))$ found in the control. This eventuality is only permitted in this of the three categories studied

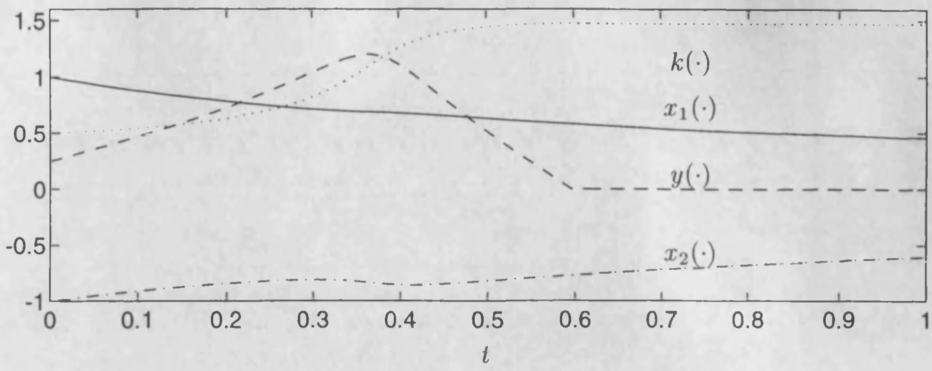


Figure 4-2: The evolution of $x(\cdot)$, $y(\cdot)$ and $k(\cdot)$.

in this chapter.

## 4.6 Exponentially stable zero-dynamics

The final subclass $\mathcal{C}_3$ of the class $\mathcal{C}$ that is stabilisable by the universal control (4.6) is classified by the following structural assumptions, **(A3.1)** to **(A3.4)**. These assumptions are similar in nature to those of the subclass inspired by Isidori and Byrnes. There is a major difference due to the Liapunov function encountered in this subclass. Note here the Liapunov function $V : \mathbb{R}^n \to \bar{\mathbb{R}}_+$ satisfies $V^+(x; f(t,x,0)) \leq -K\|x\|$ for some constant $K$, but the bound on $g(\cdot,\cdot,\cdot)$ involves any growth up to quadratic growth. Such a case did not appear in the subclass inspired by Isidori and Byrnes. We also find the approach used in the analysis of this subclass is different compared to the Isidori and Byrnes case. The subclass $\mathcal{C}_3$ of $\mathcal{C}$ is classified by the following:

**Structural Assumptions:**

**(A3.1):** The general assumptions **(G1)** to **(G4)** hold.

**(A3.2):** The function $x \mapsto \bar{f}(x)$ has the following properties:

- $\bar{f}(x) = 0 \iff x = 0$,
- $x \mapsto \bar{f}(x)$ is globally Lipschitz.
- There exists a $c > 0$ and a $K > 0$ such that for $x^0 \in \mathbb{R}^n$, the solution $x(t)$ to $\dot{x}(t) = \bar{f}(x(t))$ with $x(t_0) = x^0$ satisfies

$$\|x(t)\| \leq K \exp(-c(t-t_0))\|x^0\|, \qquad \forall t \geq t_0,$$

ie. the origin is exponentially stable.

For some known continuous function $\rho : \mathbb{R} \to \overline{\mathbb{R}}_+$ satisfying for all $r \in [0,\infty)$, $|y|^r \leq \gamma_{5,r}\rho(y)$ for all $y \in \mathbb{R}$, where $\gamma_{5,r}$ is a constant that can depend on $r$, the following hold.

**(A3.3):** The residue function, $f^*(\cdot,\cdot,\cdot)$, has the following bound:

$$\|f^*(t,x,y)\| \leq \gamma_1(1+\|x\|^{\delta_1})|y|h^*(y), \qquad \forall(t,x,y) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R},$$

where $\gamma_1 > 0$ is a possibly unknown constant and $0 \leq \delta_1 < 1$. $h^* : \mathbb{R} \to \bar{\mathbb{R}}_+$ is continuous and satisfies the following: for all $r$ such that $1 \leq r < \infty$,

$$(h^*(y))^r \leq \gamma_{2,r}\rho(y), \qquad \forall y \in \mathbb{R}$$

where $\gamma_{2,r}$ is a possibly unknown constant that can depend on $r$.

**(A3.4):** The function $g(\cdot,\cdot,\cdot)$ is in the class of functions such that

$$|g(t,x,y)| \leq \gamma_3(1+\|x\|^{\delta_2})l(y), \qquad \forall(t,x,y) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R},$$

where $\gamma_3 > 0$ is a possibly unknown constant and $0 \leq \delta_2 < 2$. The function $l : \mathbb{R} \to \bar{\mathbb{R}}_+$ satisfies for all $r$ such that $1 \leq r < \infty$,

$$(l(y))^r \leq \gamma_{4,r}\rho(y), \qquad \forall y \in \mathbb{R}$$

where $\gamma_{4,r}$ is a possibly unknown constant that can depend on $r$.

The reader will note the similarities between the assumptions given here and the assumptions imposed on the subclass $\mathcal{C}_1$. Possible examples of $h^*(\cdot)$, $l(\cdot)$ and $\rho(\cdot)$ are given in the subclass inspired by Isidori and Byrnes.

**Definition:** Let $f^o(x;w)$ denote the Clarke's derivative of a Lipschitz function $f$ at $x \in \mathbb{R}^n$ in the direction $w$, defined as follows:

$$f^o(x;w) := \limsup_{\substack{z \to x \\ h \downarrow 0}} \frac{f(z+hw)-f(z)}{h}.$$

**Definition:** Let $f^+(x;w)$ denote the following derivative of a locally Lipschitz function $f$ at $x \in \mathbb{R}^n$ in the direction $w$

$$f^+(x;w) := \limsup_{h \downarrow 0} \frac{f(x+hw)-f(x)}{h}$$

The latter derivative is used in Yoshizawa's Inverse Liapunov theorem for exponentially stable systems, see [83] and the Inverse Liapunov section in the appendix of this thesis. The following lemma gives an inequality connecting the two derivatives.

**Lemma 4.6.1** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function, then*

$$f^+(x; v+w) \leq f^+(x; v) + f^o(x; w), \qquad \forall x,\, v,\, w \in \mathbb{R}^n.$$

**Proof:**

$$\begin{aligned}
\limsup_{h \downarrow 0} \frac{f(x + h(v+w)) - f(x)}{h} &\leq \limsup_{h \downarrow 0} \frac{f(x+hv) - f(x)}{h} \\
&\quad + \limsup_{h \downarrow 0} \frac{f([x+hv] + hw) - f(x+hv)}{h} \\
&\leq \limsup_{h \downarrow 0} \frac{f(x+hv) - f(x)}{h} + \limsup_{\substack{z \to x \\ h \downarrow 0}} \frac{f(z+hw) - f(z)}{h} \\
&= f^+(x; v) + f^o(x; w)
\end{aligned}$$

□

One of the major results to be found in this subclass $C_3$, makes use of the exponential decay of the Liapunov function $V(\cdot)$ along the solution of the zero dynamics, see Theorem $B$.1.1. This property gives rise to the following lemma:

**Lemma 4.6.2** *Let $V : \mathbb{R}^n \to \bar{\mathbb{R}}_+$ be locally Lipschitz. If $x : [t_0, t) \to \mathbb{R}^n$ is a solution of (4.3) with*

$$V^+(x(t); \dot{x}(t)) + cV(x(t)) \leq \zeta(t) \leq 0, \tag{4.17}$$

*for some integrable $\zeta(\cdot)$ and some $c \geq 0$, then for some positive constants $a_0$, $a_1$ and $a_2$*

1. $\int_{t_0}^t V(x(s))\zeta(s)\, ds \leq V^2(x(t_0)) + a_0 \int_{t_0}^t \zeta^2(s)\, ds$*, for all $t \geq t_0$ and*
2. $\int_{t_0}^t V^2(x(s))\, ds \leq a_1 V^2(x(t_0)) + a_2 \int_{t_0}^t \zeta^2(s)\, ds$ *for all $t \geq t_0$.*

**Proof:** 1. Firstly note that

$$\int_{t_0}^t \exp(-cs)\zeta(s)\, ds \leq \left(\frac{1}{2c}[\exp(-2ct_0) - \exp(-2ct)]\right)^{\frac{1}{2}} \left(\int_{t_0}^t \zeta^2(s)\, ds\right)^{\frac{1}{2}}$$

by Hölder's inequality. Secondly note that

$$\begin{aligned}
&\int_{t_0}^t \zeta(s) \int_{t_0}^s \exp(-c(s-\tau))\zeta(\tau)\, d\tau\, ds \\
&\qquad \leq \left(\int_{t_0}^t \zeta^2(s)\, ds\right)^{\frac{1}{2}} \left(\int_{t_0}^t \exp(-2cs)\left[\int_{t_0}^s \exp(c\tau)\zeta(\tau)\, d\tau\right]^2 ds\right)^{\frac{1}{2}}
\end{aligned}$$

again by using Hölder's inequality. Defining

$$I(t) := \int_{t_0}^{t} \exp(-2cs) \left( \int_{t_0}^{s} \exp(c\tau)\zeta(\tau)\, d\tau \right)^2 ds$$

we get

$$\begin{aligned} I(t) &= -\frac{1}{2c} \exp(-2ct) \left( \int_{t_0}^{t} \exp(c\tau)\zeta(\tau)\, d\tau \right)^2 + \frac{1}{c} \int_{t_0}^{t} \zeta(s) \int_{t_0}^{s} \exp(-c(s-\tau))\zeta(\tau)\, d\tau\, ds \\ &\le \frac{1}{c} \left( \int_{t_0}^{t} \zeta^2(s)\, ds \right)^{\frac{1}{2}} \left( \int_{t_0}^{t} \left( \int_{t_0}^{s} \exp(-c(s-\tau))\zeta(\tau)\, d\tau \right)^2 ds \right)^{\frac{1}{2}} \end{aligned}$$

This gives

$$(I(t))^{\frac{1}{2}} \le \frac{1}{c} \left( \int_{t_0}^{t} \zeta^2(s)\, ds \right)^{\frac{1}{2}}.$$

From (4.17), we get

$$V(x(t)) \le \exp(-c(t-t_0))V(x(t_0)) + \exp(-ct) \int_{t_0}^{t} \exp(cs)\zeta(s)\, ds. \tag{4.18}$$

This gives

$$\begin{aligned} \int_{t_0}^{t} V(x(s))\zeta(s)\, ds &\le \exp(ct_0)V(x(t_0)) \int_{t_0}^{t} \exp(-cs)\zeta(s)\, ds \\ &\quad + \int_{t_0}^{t} \zeta(s) \int_{t_0}^{s} \exp(-c(s-\tau))\zeta(\tau)\, d\tau\, ds \\ &\le \frac{1}{\sqrt{(2c)}} V(x(t_0)) \left( \int_{t_0}^{t} \zeta^2(s)\, ds \right)^{\frac{1}{2}} + \frac{1}{c} \left( \int_{t_0}^{t} \zeta^2(s)\, ds \right) \\ &\le V^2(x(t_0)) + a_0 \int_{t_0}^{t} \zeta^2(s)\, ds \end{aligned}$$

for a constant $a_0 = 9/8c$. This proves part 1.

2. From (4.17) we get

$$V^2(x(t)) \le \exp(-2c(t-t_0))V^2(x(t_0)) + 2\exp(-2ct) \int_{t_0}^{t} \exp(2cs)V(x(s))\zeta(s)\, ds.$$

Integration gives

$$\begin{aligned}
\int_{t_0}^{t} V^2(x(\tau))\,d\tau &\le \exp(2ct_0)V^2(x(t_0))\left[-\frac{1}{2c}\exp(-2c\tau)\right]_{t_0}^{t} \\
&\quad + 2\int_{t_0}^{t}\exp(-2c\tau)\int_{t_0}^{\tau}\exp(2cs)V(x(s))\zeta(s)\,ds\,d\tau \\
&\le \frac{1}{2c}V^2(x(t_0)) + \left[-\frac{1}{c}\exp(-2c\tau)\int_{t_0}^{\tau}\exp(2cs)V(x(s))\zeta(s)\,ds\right]_{\tau=t_0}^{t} \\
&\quad + \frac{1}{c}\int_{t_0}^{t} V(x(\tau))\zeta(\tau)\,d\tau \\
&\le \frac{1}{2c}V^2(x(t_0)) + \frac{1}{c}\int_{t_0}^{t} V(x(\tau))\zeta(\tau)\,d\tau \\
&\le a_1 V^2(x(t_0)) + a_2\int_{t_0}^{t}\zeta^2(s)\,ds
\end{aligned}$$

by part (1) of this lemma, where $a_1 = (1+2c)/2c$ and $a_2 = a_0/c$ are constants. $\square$

We are now able to give the main theorem of this section, detailing the stability of the system (4.3) under the control (4.6). It establishes (i) the maximal interval of existence is the infinite interval $[t_0, \infty)$, (ii) the adaptive gain $k(\cdot)$ tends to a finite limit and (iii) the origin is globally attractive.

**Theorem 4.6.3** *Let $w : [t_0, \omega) \to \mathbb{R}^{n+2}$ be a solution to the differential inclusion (4.7) on its maximal interval of existence, then*

*(i)* $\omega = \infty$,

*(ii)* $\lim_{t\to\infty} k(t)$ *exists and is finite,*

*(iii)* $(x(t), y(t)) \to (0,0)$ *as* $t \to \infty$.

**Proof:** Assumption **(A3.2)** gives the existence of a Liapunov function $V(\cdot)$ that satisfies the properties of Theorem $B$.1.1 in the appendix. From Lemma 4.6.1, the fact that $V(\cdot)$ satisfies a global Lipschitz condition with Lipschitz constant $L$ and assumption **(A3.3)**, we get, for almost all $t \in [t_0, \omega)$

$$\begin{aligned}
V^+(x(t); \dot{x}(t)) &\le V^+(x(t); \bar{f}(x(t))) + V^o(x(t); f^*(t, x(t), y(t))) \\
&\le -qcV(x(t)) + L\gamma_1(1 + \|x(t)\|^{\delta_1})|y|h^*(y(t)).
\end{aligned}$$

From Young's inequality (4.9) we get $L\gamma_1|y|\|x\|^{\delta_1}h^*(y) \le c_1\|x\| + c_2(|y|h^*(y))^{r_1}$ for any constant $c_1 > 0$ with $r_1 = \frac{1}{1-\delta_1} > 1$ and $c_2 = r_1^{-1}(L\gamma_1(\delta_1^{-1}c_1)^{-\delta_1})^{r_1}$. Choosing $c_1 = \frac{1}{2}qc$ and letting $\zeta(t) = L\gamma_1|y(t)|h^*(y(t)) + c_2(|y(t)|h^*(y(t)))^{r_1}$, we get

$$V^+(x(t); \dot{x}(t)) + \frac{1}{2}qcV(x(t)) \le \zeta(t). \tag{4.19}$$

Therefore the results of Lemma 4.6.2 hold.

We now show $k(\cdot)$ is bounded, which will also give the boundedness of $y(\cdot)$. Consider the function $Y : y \mapsto \frac{1}{2}y^2$. For all $(t, w) \in \mathbb{R} \times \mathbb{R}^{n+2}$

$$\begin{aligned} y\eta &\le yg(t,x,y) + \max_{u \in \Psi(k,y)}\{yh(t,x,y)u\} \le \gamma_3(1 + \|x\|^{\delta_2})|y|l(y) + h(t,x,y)|y|\rho(y)\nu(k) \\ &\le \gamma_3|y|l(y) + \gamma_3\|x\|^{\delta_2}|y|l(y) + \mathcal{N}(k)|y|\rho(y)\gamma(t,y) \end{aligned}$$

for all $\eta \in F_2(t,w)$, where $\mathcal{N}(\cdot)$ is the Nussbaum function as defined by (4.11). Since, for all $1 \le r < \infty$, $(l(y))^r \le \gamma_{4,r}\rho(y)$ for all $y \in \mathbb{R}$

$$y\eta \le \epsilon^{-1}e_1|y|\rho(y)\gamma(t,y) + V^2(x) + \mathcal{N}(k)|y|\rho(y)\gamma(t,y)$$

where $e_1 = \gamma_3\gamma_{4,1} + q^{-1}[\gamma_3(2/\delta_2)^{-\delta_2/2}]^q(\gamma_{5,2(q-1)}\gamma_{4,2q})^{\frac{1}{2}}$ and $q = 2(2-\delta_2)^{-1}$. Therefore

$$\frac{d\|y(t)\|^2}{dt} \le \epsilon^{-1}e_1|y(t)|\rho(y(t))\gamma(t,y(t)) + V^2(x(t)) + \mathcal{N}(k(t))|y(t)|\rho(y(t))\gamma(t,y(t)).$$

So recalling $\dot{k}(t) = |y(t)|\rho(y(t))\gamma(t,y(t))$,

$$0 \le Y(y(t)) \le Y(y(t_0)) + \epsilon^{-1}e_1(k(t) - k(t_0)) + a_1V^2(x(t_0)) + a_2\int_{t_0}^{t}\zeta^2(s)\,ds + \int_{k(t_0)}^{k(t)}\mathcal{N}(k)\,dk$$

by Lemma 4.6.2. This gives

$$0 \le Y(y(t)) \le Y(y(t_0)) + a_1V^2(x(t_0)) + (\epsilon^{-1}e_1 + e_2)(k(t) - k(t_0)) + \int_{k(t_0)}^{k(t)}\mathcal{N}(k)\,dk \quad (4.20)$$

with $e_2 = a_2\epsilon^{-1}\left[(L\gamma_1)^2(\gamma_{5,2}\gamma_{2,4})^{\frac{1}{2}} + 2L\gamma_1c_2(\gamma_{5,2r_1}\gamma_{2,2(r_1+1)})^{\frac{1}{2}} + c_2^2(\gamma_{5,2(2r_1-1)}\gamma_{2,4r_1})^{\frac{1}{2}}\right]$, since $\zeta^2(t) \le e_2\dot{k}(t)$. By an analogous argument as given in Theorem 4.4.1, if $k(\cdot)$ is assumed to be unbounded, a contradiction arises. Therefore $k(\cdot)$ is bounded. Since $\mathcal{N}(\cdot)$ is continuous and $k(\cdot)$ is bounded, equation (4.20) gives $y(\cdot)$ is bounded.

Now we show that $x(\cdot)$ and therefore $w(\cdot)$ is bounded. From equation (4.19) and the fact that $\zeta(t)$ is bounded by a constant multiple of $|y(t)|\rho(y(t))$, ie.

$$\zeta(t) \le (L\gamma_1\gamma_{2,1} + c_2(\gamma_{5,2(r_1-1)}\gamma_{2,2r_1})^{\frac{1}{2}})|y(t)|\rho(y(t)),$$

$\rho(\cdot)$ being continuous and $y(\cdot)$ being bounded, we get

$$V^+(x(t);\xi) \le -\frac{1}{2}qcV(x(t)) + K$$

where $K$ is some constant. Multiplying this inequality by $\exp(\frac{1}{2}qct)$, integrating and recalling $\|x(t)\| \leq V(x(t))$ gives:

$$\|x(t)\| \leq V(x(t)) \leq \text{ const } < \infty, \qquad \forall t \in [t_0, \omega).$$

So $x(\cdot)$ is bounded, also giving the boundedness of $w(\cdot)$. This implies $\omega = \infty$ by Theorem $A.3.3$ in the appendix.

All that remains to prove is $(x(t), y(t)) \to (0,0)$ as $t \to \infty$. By the boundedness of $x(\cdot)$, $k(\cdot)$ and $y(\cdot)$, and the dynamics of $y$ in (4.7), we get $\dot{y}(\cdot) \in L_\infty$ (recall $\gamma(\mathbb{R}, K)$ is bounded for bounded $K \subset \mathbb{R}$). Also since $\dot{k}(t) = |y(t)|\rho(y(t))\gamma(t, y(t)) \geq \epsilon |y(t)|^q$, for any $q$ such that $1 \leq q < \infty$, and $k(\cdot)$ being bounded, $y(\cdot) \in L_q$. This gives $y(t) \to 0$ as $t \to \infty$ by Lemma $D.2.1$ in the appendix.

Finally we show $x(t) \to 0$ as $t \to \infty$. Let $\epsilon > 0$ be given. Let $0 \neq c \in \text{Image}(V)$ be such that $V^* = \cup_{0 \leq r \leq c} V^{-1}(r) \subset \epsilon\overline{\mathbb{B}}$, (this can be done since $V(x) \to \infty$ as $\|x\| \to \infty$). Let $\hat{\epsilon}$ be such that $\hat{\epsilon}\mathbb{B} \subset V^*$.

Recall the inequality (4.19). Since $y(t) \to 0$ as $t \to \infty$ and $\rho(\cdot)$ is continuous, there exists a time $T \geq t_0$ such that $\zeta(t) \leq L\gamma_1\gamma_{2,1}|y(t)|\rho(y(t)) + c_2\gamma_{2,r_1}|y(t)|^{r_1}\rho(y(t)) < qc\hat{\epsilon}/2$ for $t > T$. For such time,

$$V^+(x(t), \xi) < -qc\left(V(x(t)) - \frac{\hat{\epsilon}}{2}\right) \leq -qc\left(\|x(t)\| - \frac{\hat{\epsilon}}{2}\right).$$

So by Theorem $A.3.8$ in the appendix $x(t) \to V^*$ as $t \to \infty$, which implies $x(t) \to \epsilon\mathbb{B}$ as $t \to \infty$. This concludes the proof as $\epsilon > 0$ was arbitrarily small. □

### 4.6.1 Example

In this example we let $n = 2$ with the system

$$\begin{aligned}
\dot{x}_1(t) &= -x_1^3(t)/(|x_1(t)|^2 + |x_2(t)|^2) + (|x_2(t)|^{\frac{1}{2}} + 1)y^2(t)\exp(y(t)) \\
\dot{x}_2(t) &= -x_2^3(t)/(|x_1(t)|^2 + |x_2(t)|^2) + (|x_1(t)|^{\frac{1}{2}} + 1)y^2(t)\exp(y(t)) \\
\dot{y}(t) &= \sin(t)x_1(t)\exp(y^2(t)) + (2 + \sin(x_1(t)))(1 + y^2(t))u.
\end{aligned}$$

when $x \neq 0$. $\dot{x}(t) = y^2(t)\exp(y(t))[(|x_2(t)|^{1/2} + 1)\ (|x_1(t)|^{1/2} + 1)]^T$ when $x = 0$. All the controller need know about this system is that it satisfies the structural assumptions with $\rho(y) = \exp(\exp(|y|))$ and $\gamma(t, y) = 1 + y^2$. (Note the zero dynamics are asymptotically stable with a right hand side which is a homogeneous function of order 1. By [23] this is known to be a exponentially stable system.) Note that $\delta_1 = 1/2$, $h^*(y) = y\exp(y)$, $\delta_2 = 1$ and $l(y) = \exp(y^2)$. With this minimal *a priori* information, and initial conditions $x_1(0) = 1$, $x_2(0) = -1$, $y(0) = 0.5$ and $k(0) = 1$, the universal control (4.6) stabilises this system in the following way: In Figure $(4-3)$, we see the control forcing $y(t)$ to zero, letting the zero dynamics take $x(t)$ to zero.

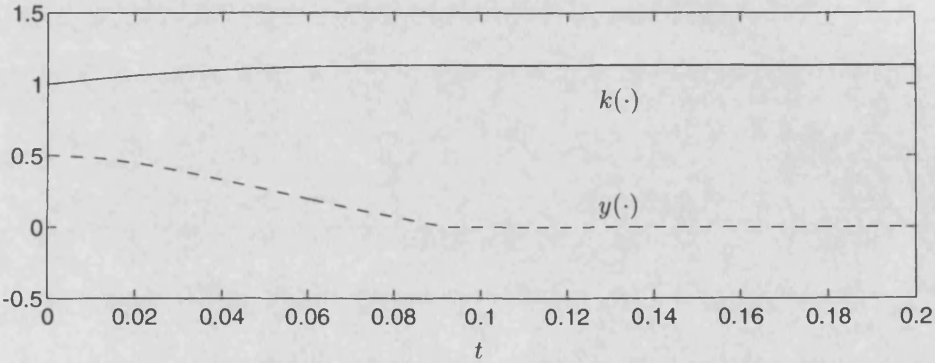


Figure 4-3: The evolution of $y(\cdot)$ and $k(\cdot)$.

# Chapter 5

# Adaptive stabilisation of multi-input nonlinear systems

## 5.1 Introduction

The work of the preceding chapter is built upon by considering a more general class of systems with multi-inputs and multi-outputs. Specifically the class of system to be considered is the class of system $\mathcal{C}$ that will have the form

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), y(t)) \\ \dot{y}(t) &= g(t, x(t), y(t)) + H(t, x(t), y(t))u \end{aligned} \tag{5.1}$$

where $x(t) \in \mathbb{R}^n$, $y(t)$, $u(t) \in \mathbb{R}^m$, $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $g : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ and $H : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{m \times m}$.

The ideas found in Chapter 3 can be extended to cope with the added difficulties arising in the more general nonlinear classes of system of this chapter. In Chapter 3, the unknown, high-frequency gain matrix was invertible and column diagonally dominant, while in this chapter the associated matrix-valued function $H(\cdot,\cdot,\cdot)$, the *input-connection matrix function*, will be both row and column diagonally dominant in the $\alpha\gamma(\cdot)$, $\beta\gamma(\cdot)$ sense, for some continuous function $\gamma(\cdot)$. This concept will be discussed in detail within the chapter, but the idea is the following: let $\alpha$, $\beta \in \mathbb{R}_+$ be such that $0 < \alpha \leq \beta$, and $\gamma : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}_+$ be a continuous function. For all $(t, x, y) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$, the property places the column and row Gerschgorin disks of the matrix $H(t, x, y)$ in the set $\{a \in \mathbb{C} \,|\, \alpha\gamma(t, y) \leq \mathrm{Re}(a) \leq \beta\gamma(t, y)\} \cup \{a \in \mathbb{C} \,|\, -\beta\gamma(t, y) \leq \mathrm{Re}(a) \leq -\alpha\gamma(t, y)\}$. This condition gives rise to $\mathcal{I}_n$, as defined in Chapter 3, being a finite unmixing set for the class of matrix-valued functions with this property, unmixing the eigenvalues of the symmetric part of the values of this function to the region between $\mathrm{Re}(z) = \alpha\gamma(t, y)$ and

$\mathrm{Re}(z) = \beta\gamma(t, y)$. An extension is made to the theory, to widen the class of input-connection matrix-valued functions to *essentially* row and column diagonally dominant matrices in the $\alpha\gamma$, $\beta\gamma$ sense. (See the subsection 'Essentially row and column diagonally dominant in the $\alpha\gamma$, $\beta\gamma$ sense'.)

Three subclasses, $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$, of the class $\mathcal{C}$ will be studied in this chapter; the same basic form as the subclasses studied in the previous chapter. Once again the Inverse Liapunov theory of [83], [23] and [42] will be used in the analysis. This theory is summarised in the appendix of this thesis. We shall find that similar structural assumptions can be imposed on the functions $f(\cdot,\cdot,0)$, $f^*(\cdot,\cdot,\cdot)$ and $g(\cdot,\cdot,\cdot)$, as those imposed in the single-input single-output case.

The universal control that will stabilise all three subclasses of $\mathcal{C}$ will have elements of the controls of Chapter 3 and the preceding chapter incorporated in its design. Once more, we use the function $K(\cdot)$, (3.4), which cycles through the elements of a finite unmixing set for matrix-valued functions which are essentially row and column diagonally dominant in the $\alpha\gamma$, $\beta\gamma$ sense. A known continuous function $\rho(\cdot)$ will also be used in the control, similar to the function of the same name found in the previous chapter. The control objective is to make the origin globally attractive under the feedback of this control.

## 5.2 A class of input-connection matrices

Here we extend the matrix theory established in Chapter 3. The ideas of row and column diagonal dominance will again be used, but in a new form. In this section we introduce the idea of *row and column diagonal dominance in the* $\alpha\gamma(\cdot)$, $\beta\gamma(\cdot)$ *sense.* Given $\alpha$ and $\beta$ with the property $0 < \alpha \le \beta$, a continuous function $\gamma : \mathbb{R}^N \to \mathbb{R}_+$ and a matrix-valued function $M : \mathbb{R}^N \to \mathbb{R}^{n\times n}$, this idea places the row and column Gerschgorin disks of the matrices $M(z)$ in the strips $\{a \in \mathbb{C} \,|\, \alpha\gamma(z) \le \mathrm{Re}(a) \le \beta\gamma(z)\}$ and $\{a \in \mathbb{C} \,|\, -\beta\gamma(z) \le \mathrm{Re}(a) \le -\alpha\gamma(z)\}$ in the complex plane, for all $z \in \mathbb{R}^N$. The set $\mathcal{I}_n$, as defined in Chapter 3, will unmix the symmetric part of the values of $M(\cdot)$, placing the eigenvalues of the symmetric part of the unmixed matrix $M(z)$ within the strip $\{a \in \mathbb{C} \,|\, \alpha\gamma(z) \le \mathrm{Re}(a) \le \beta\gamma(z)\}$ for all $z \in \mathbb{R}^N$. The theory can be extended to matrix-valued functions with the property of *essential* row and column diagonal dominance in the $\alpha\gamma$, $\beta\gamma$ sense. Such functions will be defined in this section.

### 5.2.1 Row and column diagonal dominance in the $\alpha\gamma$, $\beta\gamma$ sense

Here we give the definition of continuous matrix-valued functions with the property of row and column diagonal dominance in the $\alpha\gamma$, $\beta\gamma$ sense.

**Notation:** Given $\alpha$, $\beta \in \mathbb{R}$ with $\alpha \le \beta$, and a continuous function $\gamma : \mathbb{R}^N \to \mathbb{R}_+$, define the function $\mathcal{V}_{[\alpha,\beta,\gamma]} : \mathbb{R}^N \to \mathcal{P}(\mathbb{C})$ (where $\mathcal{P}(\mathbb{C})$ denotes the power set of the complex plane) as

$$\mathcal{V}_{[\alpha,\beta,\gamma]}(z) := \{a \in \mathbb{C} \,|\, \alpha\gamma(z) \leq \ \mathrm{Re}(a) \leq \beta\gamma(z)\}.$$

**Notation:** With $\alpha$, $\beta$ and $\gamma(\cdot)$ as above, define the function $\mathcal{V}^R_{[\alpha,\beta,\gamma]} : \mathbb{R}^N \to \mathcal{P}(\mathbb{C})$ such that $\mathcal{V}^R_{[\alpha,\beta,\gamma]}(z)$ is the reflection of $\mathcal{V}_{[\alpha,\beta,\gamma]}(z)$ in the imaginary axis for, ie. for all $z \in \mathbb{R}^N$

$$\mathcal{V}^R_{[\alpha,\beta,\gamma]}(z) := \{a \in \mathbb{C} \,|\, -\bar{a} \in \mathcal{V}_{[\alpha,\beta,\gamma]}(z)\},$$

where $\bar{a}$ denotes the complex conjugate of the complex number $a$.

**Definition:** Given $\alpha$, $\beta \in \mathbb{R}_+$ with $0 < \alpha \leq \beta$, a continuous function $\gamma : \mathbb{R}^N \to \mathbb{R}_+$ and a continuous matrix-valued function $M : \mathbb{R}^N \to \mathbb{R}^{n\times n}$, $M(\cdot)$ is row diagonally dominant in the $\alpha\gamma$, $\beta\gamma$ sense if

$$\mathcal{D}_r(M(z)) \subset \mathcal{V}_{[\alpha,\beta,\gamma]}(z) \cup \mathcal{V}^R_{[\alpha,\beta,\gamma]}(z), \qquad \forall z \in \mathbb{R}^N.$$

Similarly $M(\cdot)$ is column diagonally dominant in the $\alpha\gamma$, $\beta\gamma$ sense if

$$\mathcal{D}_c(M(z)) \subset \mathcal{V}_{[\alpha,\beta,\gamma]}(z) \cup \mathcal{V}^R_{[\alpha,\beta,\gamma]}(z), \qquad \forall z \in \mathbb{R}^N.$$

$M(\cdot)$ is said to be row and column diagonally dominant in the $\alpha\gamma$, $\beta\gamma$ sense if it is both row diagonally dominant in the $\alpha\gamma$, $\beta\gamma$ sense and column diagonally dominant in the $\alpha\gamma$, $\beta\gamma$ sense.

This class of function can have its matrix values unmixed by the set of matrices $\mathcal{I}_n$. This is proved in Chapter 3. However the way in which the set $\mathcal{I}_n$ unmixes the values of this new class of matrix-valued function is detailed in the following lemma.

**Lemma 5.2.1** *Given $\alpha$, $\beta \in \mathbb{R}_+$ with $0 < \alpha \leq \beta$, a continuous function $\gamma : \mathbb{R}^N \to \mathbb{R}_+$ and a continuous matrix-valued function $M : \mathbb{R}^N \to \mathbb{R}^{n\times n}$ that is row and column diagonally dominant in the $\alpha\gamma$, $\beta\gamma$ sense, then there exists $U \in \mathcal{I}_n$ (independent of $z$) such that*

$$\mathcal{D}_c(M(z)U) \cup \mathcal{D}_r(M(z)U) \subset \mathcal{V}_{[\alpha,\beta,\gamma]}(z), \qquad \forall z \in \mathbb{R}^N.$$

**Proof:** Let $M(z) = [m_{ij}(z)]_{n\times n}$. Since $M(\cdot)$ is row and column diagonally dominant in the $\alpha\gamma$, $\beta\gamma$ sense, for all $i$ with $1 \leq i \leq n$, either $\alpha\gamma(z) \leq m_{ii}(z) \leq \beta\gamma(z)$ for all $z \in \mathbb{R}^N$ or $-\beta\gamma(z) \leq m_{ii}(z) \leq -\alpha\gamma(z)$ for all $z \in \mathbb{R}^N$. This means for all $z \in \mathbb{R}^N$, for each $i$ the values $m_{ii}(z)$ are of fixed sign (recall $M(\cdot)$ is continuous). Therefore the matrix $U :=$ diag$\{$sign$(m_{11}(z)),\ldots,$sign$(m_{nn}(z))\} \in \mathcal{I}_n$ is independent of $z$ and therefore constant. For each $i$, either the $i^{\mathrm{th}}$ row Gerschgorin disk $\mathcal{D}_r(M(z),i) \subset \mathcal{V}_{[-\beta,-\alpha,\gamma]}(z)$ for all $z \in \mathbb{R}^N$ or $\mathcal{D}_r(M(z),i) \subset \mathcal{V}_{[\alpha,\beta,\gamma]}(z)$ for all $z \in \mathbb{R}^N$. So for each $z \in \mathbb{R}^N$ the Gerschgorin disk $\mathcal{D}_r(M(z)U,i)$ has center $|m_{ii}(z)|$ and the same radius as $\mathcal{D}_r(M(z),i)$. This gives $\mathcal{D}_r(M(z)U) \subset \mathcal{V}_{[\alpha,\beta,\gamma]}(z)$, for all $z \in \mathbb{R}^N$. A similar argument gives $\mathcal{D}_c(M(z)U) \subset \mathcal{V}_{[\alpha,\beta,\gamma]}(z)$ for all $z \in \mathbb{R}^N$. $\square$

The lemma above states that there exists an unmixing matrix $U \in \mathcal{I}_n$ independent of $z$ such that the eigenvalues of the matrix $M(z)U$ lie in the region $\mathcal{V}_{[\alpha,\beta,\gamma]}(z)$, for all $z \in \mathbb{R}^N$. The following lemma states that if, for all $z \in \mathbb{R}^N$, the row and column Gerschgorin disks of the matrix $M(z)$ lie in the region $\mathcal{V}_{[\alpha,\beta,\gamma]}(z)$, then so do the row and column Gerschgorin disks of the symmetric part of the matrix for all $z \in \mathbb{R}^N$.

**Lemma 5.2.2** *Given $\alpha$, $\beta \in \mathbb{R}_+$ with $0 < \alpha \leq \beta$, a continuous function $\gamma : \mathbb{R}^N \to \mathbb{R}_+$ and a continuous matrix-valued function $M : \mathbb{R}^N \to \mathbb{R}^{n\times n}$ that satisfies*

$$\mathcal{D}_r(M(z)) \cup \mathcal{D}_c(M(z)) \subset \mathcal{V}_{[\alpha,\beta,\gamma]}(z), \qquad \forall z \in \mathbb{R}^N, \tag{5.2}$$

*then with $N(z) := \frac{1}{2}(M(z) + M^T(z))$, the symmetric part of $M(z)$,*

$$\mathcal{D}_r(N(z)) \cup \mathcal{D}_c(N(z)) \subset \mathcal{V}_{[\alpha,\beta,\gamma]}(z), \qquad \forall z \in \mathbb{R}^N.$$

**Proof:** Let $M(z) = [m_{ij}(z)]_{n\times n}$ and $N(z) = [n_{ij}(z)]_{n\times n}$. Since $M(z)$ is real, (5.2) gives

$$m_{ii}(z) - \sum_{j=1,j\neq i}^{n} |m_{ij}(z)| \geq \alpha\gamma(z) \text{ and } m_{jj}(z) - \sum_{i=1,i\neq j}^{n} |m_{ij}(z)| \geq \alpha\gamma(z), \qquad \forall z \in \mathbb{R}^N$$

and

$$m_{ii}(z) + \sum_{j=1,j\neq i}^{n} |m_{ij}(z)| \leq \beta\gamma(z) \text{ and } m_{jj}(z) + \sum_{i=1,i\neq j}^{n} |m_{ij}(z)| \leq \beta\gamma(z), \qquad \forall z \in \mathbb{R}^N.$$

Since $N(z)$ has real entries, the centers of the row and column Gerschgorin disks are on the real line. Therefore all that must be shown is

$$n_{ii}(z) - \sum_{j=1,j\neq i}^{n} |n_{ij}(z)| \geq \alpha\gamma(z) \text{ and } n_{jj}(z) - \sum_{i=1,i\neq j}^{n} |n_{ij}(z)| \geq \alpha\gamma(z), \qquad \forall z \in \mathbb{R}^N$$

and

$$n_{ii}(z) + \sum_{j=1,j\neq i}^{n} |n_{ij}(z)| \leq \beta\gamma(z) \text{ and } n_{jj}(z) + \sum_{i=1,i\neq j}^{n} |n_{ij}(z)| \leq \beta\gamma(z), \qquad \forall z \in \mathbb{R}^N.$$

But this is true by the following arguments: for all $z \in \mathbb{R}^N$,

$$\begin{aligned}
n_{ii}(z) - \sum_{j=1,j\neq i}^{n} |n_{ij}(z)| &= m_{ii}(z) - \frac{1}{2}\sum_{j=1,j\neq i}^{n} |m_{ij}(z) + m_{ji}(z)| \\
&\geq \frac{1}{2}\left[\left(m_{ii}(z) - \sum_{j=1,j\neq i}^{n} |m_{ij}(z)|\right) + \left(m_{ii}(z) - \sum_{j=1,j\neq i}^{n} |m_{ji}(z)|\right)\right] \\
&\geq \alpha\gamma(z). \qquad (5.3)
\end{aligned}$$

Similarly for all $z \in \mathbb{R}^N$, $n_{jj}(z) - \sum_{i=1,i\neq j}^{n} |n_{ij}(z)| \geq \alpha\gamma(z)$. Also for all $z \in \mathbb{R}^N$

$$\begin{aligned}
n_{ii}(z) + \sum_{j=1,j\neq i}^{n} |n_{ij}(z)| &= m_{ii}(z) + \frac{1}{2}\sum_{j=1,j\neq i}^{n} |m_{ij}(z) + m_{ji}(z)| \\
&\leq \frac{1}{2}\left[\left(m_{ii}(z) + \sum_{j=1,j\neq i}^{n} |m_{ij}(z)|\right) + \left(m_{ii}(z) + \sum_{j=1,j\neq i}^{n} |m_{ji}(z)|\right)\right] \\
&\leq \beta\gamma(z). \qquad (5.4)
\end{aligned}$$

Similarly $n_{jj}(z) + \sum_{i=1,i\neq j}^{n} |n_{ij}(z)| \leq \beta\gamma(z)$, for all $z \in \mathbb{R}^N$, which completes the proof. □

The next result ties the two previous lemmas together giving an important inequality.

**Theorem 5.2.3** *Given $\alpha$, $\beta \in \mathbb{R}_+$ with $0 < \alpha \leq \beta$, a continuous function $\gamma : \mathbb{R}^N \to \mathbb{R}_+$ and a continuous matrix-valued function $M : \mathbb{R}^N \to \mathbb{R}^{n\times n}$ which is column and row diagonally dominant in the $\alpha\gamma$, $\beta\gamma$ sense, then there exists $U \in \mathcal{I}_n$ (independent of $z$) such that for all $z \in \mathbb{R}^N$*

$$\alpha\gamma(z)\|y\|^2 \leq \langle y, M(z)Uy\rangle \leq \beta\gamma(z)\|y\|^2, \qquad \forall y \in \mathbb{R}^n.$$

**Proof:** Since $M(\cdot)$ is row and column diagonally dominant in the $\alpha\gamma$, $\beta\gamma$ sense, Lemma 5.2.1 gives the existence of $U \in \mathcal{I}_n$ (independent of $z$) such that

$$\mathcal{D}_r(M(z)U) \cup \mathcal{D}_c(M(z)U) \subset \mathcal{V}_{[\alpha,\beta,\gamma]}(z), \qquad \forall z \in \mathbb{R}^N. \qquad (5.5)$$

By Lemma 5.2.2, with $N(z) = \frac{1}{2}(M(z)U + [M(z)U]^T)$ we get

$$\mathcal{D}_r(N(z)) \cup \mathcal{D}_c(N(z)) \subset \mathcal{V}_{[\alpha,\beta,\gamma]}(z), \qquad \forall z \in \mathbb{R}^N. \qquad (5.6)$$

For each $z$, since $N(z)$ is symmetric it has orthonormal eigenvectors $\{v_i(z)\}_{i=1,\ldots,n}$, with associated real eigenvalues $\lambda_i(z) > 0$, which form a basis of $\mathbb{R}^n$, see [19, Chapter 9, Section 10, Theorem 4]. Therefore for $y \in \mathbb{R}^n$ there exist $\gamma_i \in \mathbb{R}$ such that $y = \sum_{i=1}^{n} \gamma_i v_i(z)$. So for all

$z \in \mathbb{R}^N$,

$$\langle y, N(z)y\rangle = \sum_{i=1}^{n} \lambda_i(z)\gamma_i^2 \leq \lambda_{\max}(z)\sum_{i=1}^{n}\gamma_i^2 = \lambda_{\max}(z)\|y\|^2, \qquad \forall y \in \mathbb{R}^n$$

where $\lambda_{\max}(z) := \max_{i=1,\ldots,n}\{\lambda_i(z)\}$. and

$$\langle y, N(z)y\rangle \geq \lambda_{\min}(z)\sum_{i=1}^{n}\gamma_i^2 = \lambda_{\min}(z)\|y\|^2, \qquad \forall y \in \mathbb{R}^n$$

where $\lambda_{\min}(z) := \min_{i=1,\ldots,n}\{\lambda_i(z)\}$. By (5.6) $\lambda_{\min}(z) \geq \alpha\gamma(z)$ for all $z \in \mathbb{R}^N$ and $\lambda_{\max}(z) \leq \beta\gamma(z)$ for all $z \in \mathbb{R}^N$. By noting $\langle y, M(z)Uy\rangle = \langle y, N(z)y\rangle$, the result is obtained. □

### 5.2.2 Essentially row and column diagonal dominance in the $\alpha\gamma$, $\beta\gamma$ sense

Here the class of matrices that can be unmixed is extended to include the following:

**Definition:** Let $E_n$ be as defined in Chapter 3. Given $\alpha$, $\beta \in \mathbb{R}_+$ with $0 < \alpha \leq \beta$ and a continuous function $\gamma : \mathbb{R}^N \to \mathbb{R}_+$, the continuous matrix-valued function $M : \mathbb{R}^N \to \mathbb{R}^{n\times n}$ is said to be *essentially row and column diagonally dominant in the $\alpha\gamma$, $\beta\gamma$ sense* if there exists an $E \in E_n$ (independent of $z$), such that $z \mapsto M(z)E$ is row and column diagonally dominant in the $\alpha\gamma$, $\beta\gamma$ sense.

We now show the set of matrices $\mathcal{I}_n^E$, as defined in Chapter 3, unmixes matrix functions that are essentially row and column diagonally dominant in the $\alpha\gamma$, $\beta\gamma$ sense, providing a similar inequality as found in Theorem 5.2.3.

**Theorem 5.2.4** *Given $\alpha$, $\beta \in \mathbb{R}_+$ with $0 < \alpha \leq \beta$, a continuous function $\gamma : \mathbb{R}^N \to \mathbb{R}_+$ and a continuous matrix-valued function $M : \mathbb{R}^N \to \mathbb{R}^{n\times n}$ that is essentially column and row diagonally dominant in the $\alpha\gamma$, $\beta\gamma$ sense, then there exists a $G \in \mathcal{I}_n^E$ (independent of $z$) such that for all $z \in \mathbb{R}^N$*

$$\alpha\gamma(z)\|y\|^2 \leq \langle y, M(z)Gy\rangle \leq \beta\gamma(z)\|y\|^2, \qquad \forall y \in \mathbb{R}^n.$$

**Proof:** Since $M(\cdot)$ is essentially row and column diagonally dominant in the $\alpha\gamma$, $\beta\gamma$ sense, there exists an $E \in E_n$ (independent of $z$) such that $z \mapsto M(z)E$ is row and column diagonally dominant in the $\alpha\gamma$, $\beta\gamma$ sense. By Theorem 5.2.3 there exists a $U \in \mathcal{I}_n$ (independent of $z$) such that for all $z \in \mathbb{R}^N$,

$$\alpha\gamma(z)\|y\|^2 \leq \langle y, M(z)EUy\rangle \leq \beta\gamma(z)\|y\|^2, \qquad \forall y \in \mathbb{R}^n.$$

Clearly $G := EU \in \mathcal{I}_n^E$ (independent of $z$) is the required element to conclude the proof. □

## 5.3 A class of multi-input, multi-output nonlinear systems

The general class $\mathcal{C}$ of nonlinear system to be studied is of the following form

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), y(t)) \\ \dot{y}(t) &= g(t, x(t), y(t)) + H(t, x(t), y(t))u \end{aligned} \tag{5.7}$$

where $x(t) \in \mathbb{R}^n$, $y(t)$, $u(t) \in \mathbb{R}^m$, $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $g : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ and $H$ : $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{m \times m}$. (Note, if $f$, $g$ and $H$ were time invariant this would be the multi-input, multi-output case of the Isidori normal form, see [39] and [11].) Similar structural assumptions will be made on this class of system, as those made on the class of system (4.3). However the notable exception, **(G2)**, comes about as a result of allowing general multi-dimension in the input and output. Here the ideas of the preceding section are used. The general structural assumptions are given as the following:

**General Structural Assumptions.**

**(G1):** The functions $f(\cdot, \cdot, \cdot)$ and $g(\cdot, \cdot, \cdot)$ are continuous, and the matrix-valued function $H(\cdot, \cdot, \cdot)$ is continuous.

**(G2):** There exists an $\epsilon > 0$, a known continuous function $\gamma : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}_+$ with $\gamma(t, y) \geq \epsilon$ for all $(t, y) \in \mathbb{R} \times \mathbb{R}^m$, and scalars $\alpha$, $\beta \in \mathbb{R}_+$ (not necessarily known) with $0 < \alpha \leq \beta < \infty$ such that

(i) the matrix-valued function $H(\cdot, \cdot, \cdot)$ is essentially row and column diagonally dominant in the $\alpha\hat{\gamma}(\cdot, \cdot, \cdot)$, $\beta\hat{\gamma}(\cdot, \cdot, \cdot)$ sense, where $\hat{\gamma}(t, x, y) = \gamma(t, y)$ for all $(t, x, y) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$,

(ii) $K \subset \mathbb{R}^m$ bounded $\implies \gamma(\mathbb{R}, K)$ bounded.

**(G3):** The point $(x, y) = (0, 0)$ is the unique stationary point of the system (5.7) in the absence of control.

**(G4):** The function $f(\cdot, \cdot, 0)$ is time invariant, ie. $f(t, x, 0) = \bar{f}(x)$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$ for some continuous function $\bar{f}(\cdot)$.

Structural assumption **(G3)** makes the origin the target of the universal control of the next section. **(G1)** is imposed for analytical reasons, so the existence of a solution will be guaranteed. **(G4)** is imposed in order that inverse Liapunov functions for what will be a stable system will be time invariant. Structural assumption **(G2)** provides information of how the control will affect the system. This condition also gives rise to the set $I_m^E$ being a known finite unmixing set of the matrix-valued function $H(\cdot, \cdot, \cdot)$.

Again we look at three different but non-exclusive subclasses, $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$, of the class $\mathcal{C}$. These will be essentially the same as those considered in Chapter 4, but in this multi-input, multi-output case. The ideas met in this previous chapter will play a prominent role in the analysis of the three subclasses of $\mathcal{C}$.

Once again the Inverse Liapunov theory of [83], [23] and [42], reproduced in the appendix, will be used in the analysis of the subclasses.

## 5.4 A universal control

The universal control that makes the origin globally attractive for subclasses of $\mathcal{C}$, is based on the universal control of the preceding chapter. This is due to the similarities between the systems (4.3) and (5.7). The main difference is the introduction of a cycling mechanism which cycles through the unmixing set $I_m^E$ in the same manner as found in Chapter 3. Of course if **(G2)** were replaced with:

**(G2-optional):** There exists an $\epsilon > 0$, a known continuous function $\gamma : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}_+$ with $\gamma(t, y) \geq \epsilon$ for all $(t, y) \in \mathbb{R} \times \mathbb{R}^m$, and scalars $\alpha$, $\beta \in \mathbb{R}_+$ (not necessarily known) with $0 < \alpha \leq \beta < \infty$ such that

(i) the matrix-valued function $H(\cdot,\cdot,\cdot)$ is row and column diagonally dominant in the $\alpha\hat{\gamma}(\cdot,\cdot,\cdot)$, $\beta\hat{\gamma}(\cdot,\cdot,\cdot)$ sense where $\hat{\gamma}(t, x, y) = \gamma(t, y)$ for all $(t, x, y) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$,

(ii) $K \subset \mathbb{R}^m$ bounded $\implies$ $\gamma(\mathbb{R}, K)$ bounded.

then $I_m$ (a smaller set of matrices) can be used as the unmixing set. The cycling strategy below would then only have to cycle through this smaller set of matrices.

### 5.4.1 Cycling strategy

In the same way as in Chapter 3, see [70] and [73], define the sequence $(\tau_n)$, $\tau_{n-1} < \tau_n$ such that

$$\lim_{n\to\infty} \tau_n = \infty \text{ and } \lim_{n\to\infty} \tau_{n-1}/\tau_n = 0.$$

Label the $r := 2^n n!$ elements of the set $I_n^E$ as $K_1$ to $K_r$. With the same notation as Chapter 3, define the cycling strategy, between the elements of this finite unmixing set, as

$$K(s) := \begin{cases} K_1, & s \in (\infty, \tau_1] \\ \dots & \dots \\ K_i + \xi_i(s)(K_{i+1} - K_i), & s \in T_i,\ i \in \{1, 2, \dots, r-1\} \\ \dots & \dots \\ K_r + \xi_r(s)(K_1 - K_r), & s \in T_r. \end{cases} \tag{5.8}$$

See Chapter 3 for more information on this cycling strategy.

### 5.4.2 The control

The universal control takes on elements from the approach of the preceding chapter and Chapter 3. We formally give the control as:

$$\begin{aligned} u(t) &= -k(t)\rho(y(t))\|y(t)\|^{-1}K(k(t))y(t) \\ \dot{k}(t) &= \|y(t)\|\rho(y(t))\gamma(t,y(t)), \qquad k(0) \in \mathbb{R}_+, \end{aligned} \tag{5.9}$$

where $\rho(\cdot)$ is a known continuous function that is dependent on the subclass of $\mathcal{C}$ and $\gamma(\cdot,\cdot)$ is the known continuous function of **(G2)**. However, the control $u$ has a discontinuity at the point $y = 0$. Again the technique presented in [70], and used in previous chapters, embeds the control in the following inclusion.

$$\begin{aligned} u(t) &\in \Psi(y(t),k(t)) := -k(t)\rho(y(t))K(k(t))\psi(y(t)) \\ \dot{k}(t) &= \|y(t)\|\rho(y(t))\gamma(t,y(t)), \qquad k(t_0) \in \mathbb{R}_+. \end{aligned} \tag{5.10}$$

where $\psi(\cdot)$ is the set-valued map (2.7). $\psi(\cdot)$ is an upper-semicontinuous set-valued map, with convex and compact values.

Therefore the class of systems $\mathcal{C}$ can be embedded in the following differential inclusion: Let $w = (x, y, k)$,

$$\dot{w}(t) \in F(t,w(t)), \qquad w(t_0) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_+ \tag{5.11}$$

with $F(t,w) := F_1(t,w) \times F_2(t,w) \times F_3(t,w)$, where

$$\begin{aligned} F_1(t,w) &:= \{f(t,x,y)\} \\ F_2(t,w) &:= \{g(t,x,y) + H(t,x,y)u \mid u \in \Psi(y,k)\} \\ F_3(t,w) &:= \{\|y\|\rho(y)\gamma(t,y)\}. \end{aligned}$$

By Theorem $A.3.1$ and Theorem $A.3.2$ there exists a solution to this initial value problem on its maximal interval of existence.

## 5.5 A subclass inspired by Byrnes and Isidori

The three subclasses of $\mathcal{C}$ to be considered are similar to those of the previous chapter. We begin with the subclass $\mathcal{C}_1$ of the class $\mathcal{C}$ inspired by the example given in [9], see Chapter 4 for details. Here we shall show that the universal control (5.10) stabilises the multi-input, multi-output generalisation of this subclass.

We find the structural assumptions of this multi-dimensional extension are similar in nature to those of the previous chapter, since the main difference is just the increase in dimension of the input and output. The structural assumptions are the following:

**Structural Assumptions:**

**(A1.1):** The general assumptions, **(G1)**, **(G2)**, **(G3)** and **(G4)** hold.

**(A1.2):** The following form of stability is imposed on the zero dynamics of the system (5.7), $\dot{x}(t) = f(t, x(t), 0) = \bar{f}(x(t))$: there exists a possibly unknown $C^1$ function $W : \mathbb{R}^n \to \overline{\mathbb{R}}_+$ with the properties

(i) $W(0) = 0$,

(ii) $W(x) > 0$ for $x \neq 0$ and $W(x) \to \infty$ as $\|x\| \to \infty$,

(iii) $\langle \nabla W(x), \bar{f}(x) \rangle \leq -\gamma_0 \|x\|^{a_1}$, for all $x \in \mathbb{R}^n$ where $\gamma_0 > 0$ and $a_1 > 1$ are possibly unknown,

(iv) $\|\nabla W(x)\| < \gamma_1 \|x\|^{a_1 - a_2}$ for all $x \in \mathbb{R}^n$, where $0 < a_2 < a_1$ and $\gamma_1 > 0$ are possibly unknown.

For some known continuous function $\rho : \mathbb{R}^m \to \bar{\mathbb{R}}_+$ satisfying for all $r \in [0, \infty)$, $\|y\|^r \leq \gamma_{6,r} \rho(y)$ for all $y \in \mathbb{R}^m$, where $\gamma_{6,r}$ is a (possibly unknown) constant that can depend on $r$, the following hold.

**(A1.3):** The residue function belongs to the class of functions with the following bound:

$$\|f^*(t, x, y)\| \leq \gamma_2 \|y\| h^*(y)(1 + \|x\|^{\delta_1}), \qquad \forall (t, x, y) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$$

for some possibly unknown $\gamma_2 > 0$ and $0 \leq \delta_1 < a_2$, with $a_2$ as in **(A1.2)**. $h^* : \mathbb{R}^m \to \bar{\mathbb{R}}_+$ is a continuous function satisfying, for all $r$ such that $1 \leq r < \infty$,

$$(h^*(y))^r \leq \gamma_{3,r} \rho(y), \qquad \forall y \in \mathbb{R}^m,$$

where $\gamma_{3,r}$ is a constant which can depend on $r$.

**(A1.4):** The function $g(\cdot, \cdot, \cdot)$ is a continuous function belonging to the class of functions with the following bound:

$$\|g(t, x, y)\| \leq \gamma_4 l(y)(1 + \|x\|^{\delta_2}), \qquad \forall (t, x, y) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$$

for $\gamma_4 > 0$ and $0 \leq \delta_2 < a_1$, possibly unknown, with $a_1$ as in **(A1.2)**. $l : \mathbb{R}^m \to \bar{\mathbb{R}}_+$ is a function satisfying, for all $r$ such that $1 \leq r < \infty$,

$$(l(y))^r \leq \gamma_{5,r} \rho(y), \qquad \forall y \in \mathbb{R}^m$$

where $\gamma_{5,r}$ is a constant that can depend on $r$.

As before **(A1.2)** and **(A1.3)** are automatically satisfied if $f(\cdot,\cdot,\cdot)$ is a positively-homogeneous function of order $k > 1$ in its dependence on its second and third arguments, $f^*(\cdot,\cdot,\cdot)$ is continuously differentiable and $f^*(\mathbb{R}, \mathbb{S}^{n+m-1})$ is bounded.

As before an important analytical consideration is the bound in **(A1.3)** is divisible by $\|y\|$. Also the bound with respect to $x$ is stronger in **(A1.3)** than in **(A1.4)**. See Chapter 4 for details.

An important inequality involving the cycling function $K(\cdot)$, between the finite number of elements of the unmixing set $I_m^E$, is given. This inequality appears in all the subclasses of the class $C$. It gives the intervals of $\mathbb{R}$, on which a bound on an inner product is non-positive. The interval is such that the value of the function $K(\cdot)$ is 'near' to the element of the set $I_m^E$ which unmixes the values of matrix-valued function $H(\cdot,\cdot,\cdot)$.

**Lemma 5.5.1** *Let $\alpha$, $\beta \in \mathbb{R}_+$ be such that $0 < \alpha \leq \beta$, let $\gamma : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}_+$ and $H(\cdot,\cdot,\cdot) : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{m \times m}$ a continuous, matrix-valued function that is essentially row and column diagonally dominant in the $\alpha\gamma(\cdot,\cdot)$, $\beta\gamma(\cdot,\cdot)$ sense, and let $K(\cdot)$ be the cycling function* (5.8)*, then there exists an $\epsilon^* \in (0,1]$ and $j \in \{1,\ldots,r\}$ such that with $I_{\epsilon^*,j}$ defined as in Chapter 3, $(I_{\epsilon^*,j} = \cup_{q\in\mathbb{N}_0}[(1-\epsilon^*)\tau_{rq+j} - \epsilon^*\tau_{rq+j-1}, \epsilon^*\tau_{rq+j+1} + (1-\epsilon^*)\tau_{rq+j}])$,*

$$s \in I_{\epsilon^*,j} \implies -\langle y, H(t,x,y)K(s)y\rangle \leq -\frac{1}{2}\alpha\gamma(t,y)\|y\|^2, \qquad \forall (t,x,y) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m,$$

*and*

$$s \notin I_{\epsilon^*,j} \implies -\langle y, H(t,x,y)K(s)y\rangle \leq \frac{1}{2}\alpha\gamma^*\gamma(t,y)\|y\|^2, \qquad \forall (t,x,y) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m,$$

*where $\gamma^* = 4\sqrt{m}\beta/\alpha$.*

**Proof:** By Theorem 5.2.4 there exists a $j \in \{1,\ldots,r\}$ such that

$$\alpha\gamma(t,y)\|y\|^2 \leq \langle y, H(t,x,y)K_j y\rangle \leq \beta\gamma(t,y)\|y\|^2, \qquad \forall (t,x,y) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m.$$

Define $\epsilon^* := \min\{\alpha/4\sqrt{m}\beta, 1\}$. For $s \in I_{\epsilon^*,j}$,

$$-\langle y, H(t,x,y)K(s)y\rangle = \begin{cases} -\langle y, H(t,x,y)K_j y\rangle & \\ \quad +(1-\xi_{j-1}(s))\langle y, H(t,x,y)(K_j - K_{j-1})y\rangle, & s \in T_{j-1} \cap I_{\epsilon^*,j} \\ -\langle y, H(t,x,y)K_j y\rangle & \\ \quad -\xi_{j-1}(s)\langle y, H(t,x,y)(K_{j+1} - K_j)y\rangle, & s \in T_j \cap I_{\epsilon^*,j} \end{cases}$$

$$\leq -(\alpha - 2\sqrt{m}\epsilon^*\beta)\gamma(t,y)\|y\|^2 \leq -\frac{1}{2}\alpha\gamma(t,y)\|y\|^2$$

by the choice of $\epsilon^*$, and since $\|H(t,x,y)\| \le \sqrt{m}\beta\gamma(t,y)$ for all $(t,x,y) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$. (The latter inequality is obtained by recalling the Schur matrix norm [80],

$$\|A\|_2 \le \|A\|_S := \left(\sum_{i,j} |a_{ij}|^2\right)^{\frac{1}{2}},$$

where $A := [a_{ij}]$. By the property on $H(t,x,y) = [h_{ij}(t,x,y)]$, $\sum_{i=1}^n |h_{ij}(t,x,y)| \le \beta\gamma(t,y)$ for all $(t,x,y) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$. Therefore for each $j$, $\sum_{i=1}^n |h_{ij}(t,x,y)|^2 \le (\beta\gamma(t,y))^2$.)

For general $s$

$$-\langle y, H(t,x,y)K(s)y\rangle \le 2\|H(t,x,y)\|\|y\|^2 \le \frac{1}{2}\alpha\gamma^*\gamma(t,y)\|y\|^2, \qquad \forall (t,x,y) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m,$$

where $\gamma^* := 4\sqrt{m}\beta/\alpha$. This concludes the proof. □

Now we give the analysis that establishes (i) the maximal interval of existence is the infinite interval $[t_0, \infty)$, (ii) the adaptive gain $k(\cdot)$ tends to a finite limit and (iii) the origin is globally attractive with respect to the solutions $x(t)$ and $y(t)$. This result is given in the following theorem:

**Theorem 5.5.2** *Let $w : [t_0, \omega) \to \mathbb{R}^{n+m+1}$ be a solution to the differential inclusion (5.11), on its maximal interval of existence, then*

*(i) $\omega = \infty$,*

*(ii) $\lim_{t\to\infty} k(t)$ exists and is finite,*

*(iii) $(x(t), y(t)) \to (0,0)$ as $t \to \infty$.*

**Proof:** As with the single-input, single-output case, Young's inequality plays a prominent role in the proof. The form that is useful, is that given in (4.9).

In a similar manner to Theorem 4.4.1, define $V(x,y) := W(x) + \frac{1}{2}\|y\|^2$ where $W(\cdot)$ is the function in **(A1.2)**. For all $\eta \in F_1(t,w) \times F_2(t,w)$, $y \neq 0$,

$$\langle \nabla V(x,y), \eta\rangle \le -\frac{1}{2}\gamma_0\|x\|^{a_1} + c^\dagger\|y\|\rho(y) - k\rho(y)\|y\|^{-1}\langle y, H(t,x,y)K(k)y\rangle$$

where $c^\dagger$ is a constant obtained from **(A1.3)** to **(A1.4)**. Defining $\epsilon^*$, $\gamma^*$, $j$ and $I_{\epsilon^*,j}$ as in Lemma 5.5.1 and $\nu : \mathbb{R} \to \{\alpha/2, -\alpha\gamma^*/2\}$ as

$$\nu(k) \mapsto \begin{cases} \alpha/2, & k \in I_{\epsilon^*,j} \\ -\alpha\gamma^*/2, & k \notin I_{\epsilon^*,j} \end{cases} \tag{5.12}$$

gives

$$\langle \nabla V(x,y), \eta \rangle \leq -\frac{1}{2}\gamma_0 \left(\|x\|^{a_1} + \|y\|^2\right) + \mu\|y\|\rho(y)\gamma(t,y) - k\nu(k)\rho(y)\gamma(t,y)\|y\|. \quad (5.13)$$

where $\mu := \epsilon^{-1}(c^\dagger + \gamma_0\gamma_{6,1})/2$ by Lemma 5.5.1. Therefore

$$\begin{aligned}\frac{dV(x(t),y(t))}{dt} &\leq -\frac{1}{2}\gamma_0 \left(\|x(t)\|^{a_1} + \|y(t)\|^2\right) + \mu\|y(t)\|\rho(y(t))\gamma(t,y(t)) \\ &\quad - k(t)\nu(k(t))\rho(y(t))\gamma(t,y(t))\|y(t)\|. \quad (5.14)\end{aligned}$$

Recall that $\dot{k}(t) = \|y(t)\|\gamma(t,y)\rho(y(t))$, with (5.14) this gives

$$\begin{aligned}0 \leq W(x(t)) + \frac{1}{2}\|y(t)\|^2 &\leq W(x(t_0)) + \frac{1}{2}\|y(t_0)\|^2 \quad (5.15) \\ &\quad + \mu(k(t) - k(t_0)) - \int_{k(t_0)}^{k(t)} k\nu(k)\, dk.\end{aligned}$$

Suppose for a contradiction $k(\cdot)$ is unbounded. Since $\dot{k}(t) \geq 0$, we can assume the existence of a $\tau > t_0$ such that for all $t > \tau$, $k(t) \geq 1$. This and (5.15) gives

$$0 \leq c^* - \limsup_{k\to\infty} \frac{1}{k}\int_{k(t_0)}^{k} k\nu(k)\, dk$$

where $c^* := \mu + W(x(t_0)) + \|y(t_0)\|^2/2$. This is a contradiction by Lemma 3.5.2. Hence the boundedness of $k(\cdot)$.

From the boundedness of $k(\cdot)$ and (5.15) the boundedness of $x(\cdot)$ and $y(\cdot)$ are established. This in turn, by Theorem $A$.3.3 in the appendix, gives $\omega = \infty$.

It remains to prove $(x(t), y(t)) \to (0,0)$ as $t \to \infty$. This is done by defining

$$V^*(x,y,k) := V(x,y) - \left(\mu k - \int_{k(t_0)}^{k} k\nu(k)\, dk\right),$$

and noting, for $\eta \in F_1(t,w) \times F_2(t,w) \times F_3(t,w)$,

$$\frac{d}{dt}V^*(w(t)) \leq -\frac{1}{2}\gamma_0(\|x(t)\|^{a_1} + \|y(t)\|^2).$$

By Theorem $A$.3.7 in the appendix, $(x(t), y(t)) \to (0,0)$ as $t \to \infty$. □

### 5.5.1 Example

Here we consider the following system, $n = 2$ and $m = 2$, which satisfies the structural assump-

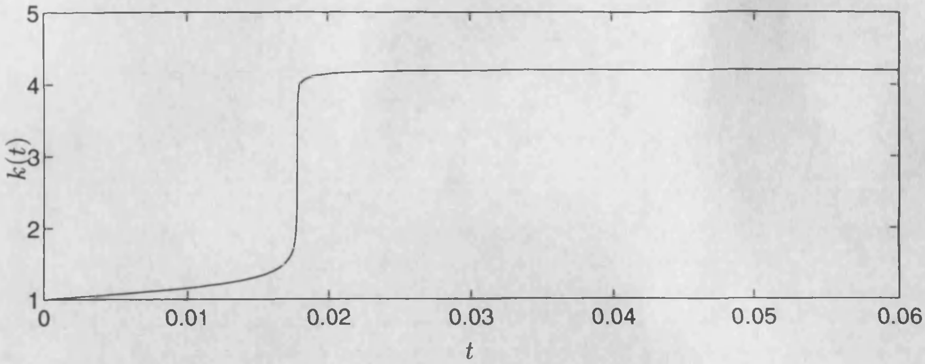


Figure 5-1: The evolution of $k(\cdot)$.

tions of this section with $\rho(y) = \exp(\|y\|)$ and $\gamma(t,y) = \exp(\|y\|)$.

$$\dot{x}(t) = -\begin{bmatrix} x_1^3(t) \\ x_2^3(t) \end{bmatrix} + 50 \begin{bmatrix} \langle y(t), x(t)\rangle\, x_2(t) \\ \langle y(t), [|x_1(t)|^{1.9}\ x_2(t)]^T\rangle x_1(t) \end{bmatrix}$$
$$\dot{y}(t) = (x_1(t)x_2(t))^2 \begin{bmatrix} \cos(t)y_1(t)x_1(t) \\ \sin(t)y_2(t)x_2(t) \end{bmatrix} + \begin{bmatrix} -4\exp(\|y(t)\|) & y_1(t)\sin(t) \\ y_2(t)\cos(t) & 4\exp(\|y(t)\|) \end{bmatrix} u.$$

The matrix-valued function

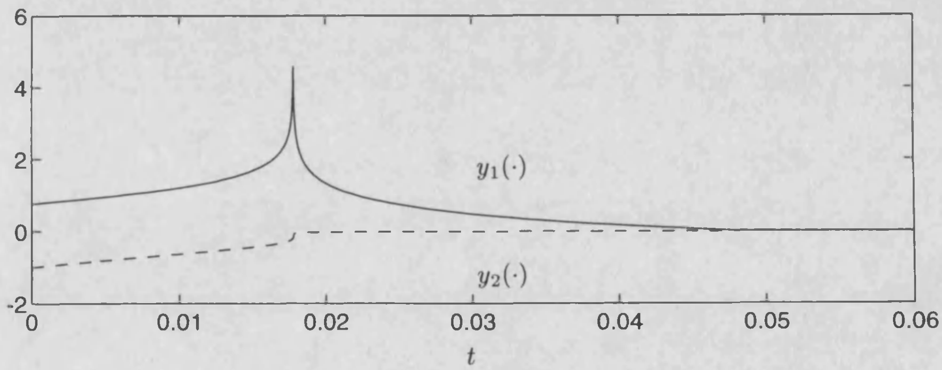


Figure 5-2: The evolution of $y(\cdot)$.

$$(t,y) \mapsto \begin{bmatrix} -4\exp(\|y\|) & y_1\sin(t) \\ y_2\cos(t) & 4\exp(\|y\|) \end{bmatrix}$$

is row and column diagonally dominant in the $(t,y) \mapsto 3\exp(\|y\|)$, $(t,y) \mapsto 5\exp(\|y\|)$ sense, and the zero dynamics are asymptotically stable with a positive homogeneous right hand side of order 3 (see Chapter 4). Note that $\|f^*(t,x,y)\| \le 50\|y\|(1+\|x\|^{2.9})$ and $\|g(t,x,y)\| \le \|y\|(1+\|x\|^5)$ for all $(t,x,y) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$. The power of 5 in the bound on $g(\cdot,\cdot,\cdot)$ is permitted, because the positively homogeneous function theory states that $a_1$ in **(A1.4)** can be arbitrarily large. With $x(0) = [2\ \ -2]$, $y(0) = [0.75\ \ -1]$ and $k(0) = 1$, the following results, see Figures $(5-1)$,

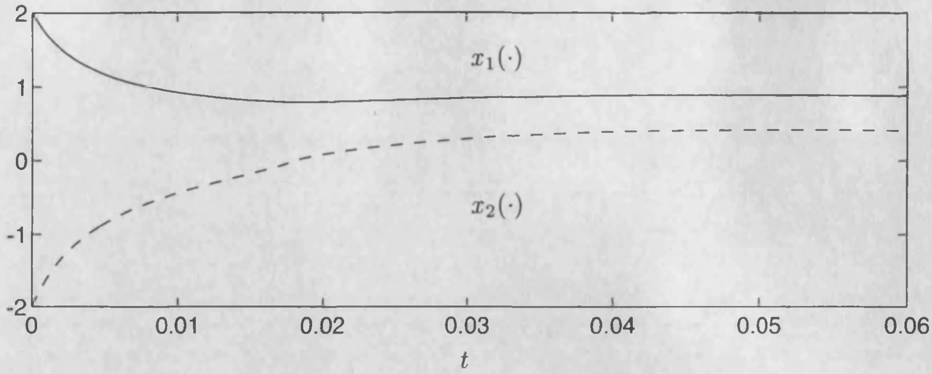


Figure 5-3: The evolution of $x(\cdot)$.

$(5-2)$ and $(5-3)$ are obtained:

In Figure $(5-1)$ we see the usual evolution of $k(\cdot)$, similar to that of previous examples, levelling out around $t = 0.02$. After this time the output tends to the origin as seen in Figure $(5-2)$.

In Figure $(5-3)$, we see the evolution of the state dynamics during this time. After this time the zero dynamics take the state to the origin, but slowly since the zero dynamics are $\dot{x}(t) = -[x_1^3(t)\ x_2^3(t)]^T$.

## 5.6 A system with a bounded-input/bounded-output subsystem

The second subclass $\mathcal{C}_2$ of the class $\mathcal{C}$ is the multi-dimensional case of the second subclass of (4.3) looked at in the previous chapter. The universal control (5.10) makes the origin globally attractive. The structural assumptions are the following.

**Structural Assumptions:**

**(A2.1):** The general assumptions **(G1)** to **(G4)** hold.

**(A2.2):** Bounded input (i.e. $\|y\|_\infty < \infty$) for the system $\dot{x}(t) = f(t, x(t), y(t))$ implies bounded output (i.e. $\|x\|_\infty < \infty$).

**(A2.3):** For some known continuous function $\rho : \mathbb{R}^m \to \bar{\mathbb{R}}_+$ satisfying $\rho(y) \geq \|y\|^p$ for all $y \in \mathbb{R}^m$ for some $0 \leq p < \infty$, the function $g(\cdot, \cdot, \cdot)$ is in the class of functions with the following bound:

$$\|g(t, x, y)\| \leq \gamma_1 \rho(y), \qquad \forall (t, x, y) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m,$$

where $\gamma_1 > 0$ is a possibly unknown constant.

**(A2.4):** The function $x \mapsto f(t, x, 0) = \bar{f}(x)$ has the following properties:

- $x = 0 \iff \bar{f}(x) = 0$.
- The origin is globally asymptotically stable for the system $\dot{x}(t) = \bar{f}(x(t))$.
- $x \mapsto \bar{f}(x)$ is locally Lipschitz.

**(A2.5):** There exists a continuous function $f^\dagger : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}_+$ with $f^\dagger(x, 0) = 0$ for all $x \in \mathbb{R}^n$, such that

$$\|f^*(t,x,y)\| \leq f^\dagger(x,y), \qquad \forall (t,x,y) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m.$$

The aim of this subsection is to show the origin is globally attractive for the solutions $x(t)$ and $y(t)$, under the effects of the universal control (5.10).

The Proposition 4.5.1 of Chapter 4 will be used in the analysis of the main result of this section.

The main theorem of this section establishes the following: (i) the maximal interval of existence of the solution of the differential inclusion (5.11) is the infinite interval $[t_0, \infty)$, (ii) the adaptive gain $k(\cdot)$ tends to a finite limit and (iii) the origin is globally attractive with respect to the solutions $x(t)$ and $y(t)$.

**Theorem 5.6.1** *Let $w : [t_0, \omega) \to \mathbb{R}^{n+m+1}$ be a solution to the differential inclusion (5.11) on its maximal interval of existence, then*

*(i) $\omega = \infty$,*

*(ii) $\lim_{t\to\infty} k(t)$ exists and is finite,*

*(iii) $(x(t), y(t)) \to (0,0)$ as $t \to \infty$.*

**Proof:** As with Theorem 4.5.2, we begin by showing the boundedness of $k(\cdot)$. This in turn gives the boundedness of $y(\cdot)$. Consider the function $Y : y \mapsto \frac{1}{2}\|y\|^2$. For all $(t,w) \in \mathbb{R} \times \mathbb{R}^{n+m+1}$, $y \neq 0$,

$$\begin{aligned}
\langle y, \eta \rangle &\leq \langle y, g(t,x,y) \rangle + \max_{u \in \Psi(y,k)} \{\langle y, H(t,x,y)u \rangle\} \\
&\leq \gamma_1 \|y\| \rho(y) - k\rho(y)\|y\|^{-1} \langle y, H(t,x,y)K(k)y \rangle \\
&\leq \epsilon^{-1}\gamma_1 \|y\|\rho(y)\gamma(t,y) - k\rho(y)\nu(k)\gamma(t,y)\|y\|
\end{aligned}$$

for all $\eta \in F_2(t,w)$, by Lemma 5.5.1, where $\nu(\cdot)$ is defined as in (5.12). Therefore

$$\frac{d\|y(t)\|^2}{dt} \leq \epsilon^{-1}\gamma_1 \|y(t)\|\rho(y(t))\gamma(t,y(t)) - k(t)\rho(y(t))\nu(k(t))\gamma(t,y(t))\|y(t)\|.$$

Recalling $\dot{k}(t) = \|y(t)\|\rho(y(t))\gamma(t,y)$, gives

$$0 \leq \frac{1}{2}\|y(t)\|^2 \leq \frac{1}{2}\|y(t_0)\|^2 + \epsilon^{-1}\gamma_1 (k(t) - k(t_0)) + \int_{k(t_0)}^{k(t)} k\nu(k)\, dk. \tag{5.16}$$

By an analogous argument as found in Theorem 5.5.2, by assuming $k(\cdot)$ is unbounded, a contradiction arises. Therefore $k(\cdot)$ is bounded.

Since $k(\cdot)$ is bounded, from (5.16) the boundedness of $y(\cdot)$ is established. From assumption **(A2.2)** the boundedness of $x(\cdot)$ is also established. This in turn gives $\omega = \infty$ by Theorem $A$.3.3 in the appendix.

Now we show $y(t) \to 0$ as $t \to \infty$. This is true by an analogous argument as in Theorem 4.5.2. All that remains to be proved is $x(t) \to 0$ as $t \to \infty$. This is true by Proposition 4.5.1 in the previous chapter. □

### 5.6.1 Example

In this section we consider the system with $n = 2$ and $m = 2$:

$$\begin{aligned}
\dot{x}(t) &= -\begin{bmatrix} x_1^3(t) \\ x_2^3(t) \end{bmatrix} + 50\|y(t)\| \begin{bmatrix} \sin(10t)x_1(t) \\ \cos(10t)x_2(t) \end{bmatrix} \\
\dot{y}(t) &= \begin{bmatrix} \cos(t)y_1^2(t) \\ \sin(t)y_2^2(t) \end{bmatrix} + \begin{bmatrix} 4(1+\|y(t)\|) & -\sin(t) \\ \cos(t) & -4(1+\|y(t)\|) \end{bmatrix} u.
\end{aligned}$$

Note that the matrix-valued function

$$(t, y) \mapsto \begin{bmatrix} 4(1+\|y\|) & -\sin(t) \\ \cos(t) & -4(1+\|y\|) \end{bmatrix}$$

is row and column diagonally dominant in the $(t, y) \mapsto 3(1 + \|y\|)$, $(t, y) \mapsto 5(1 + \|y\|)$ sense.

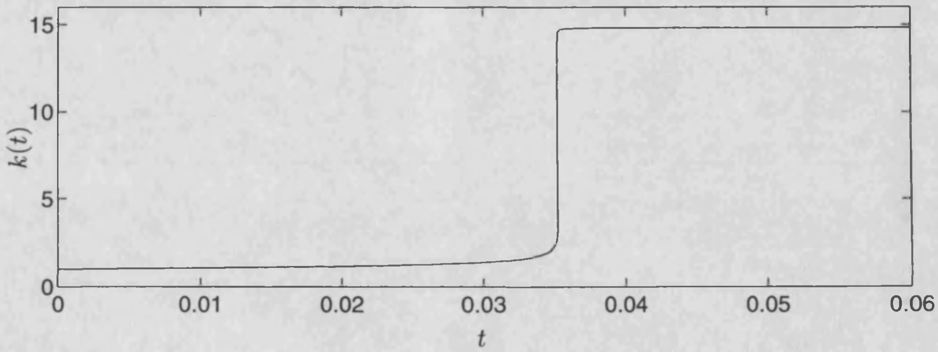


Figure 5-4: The evolution of $k(\cdot)$.

All the controller need know is that the structural assumptions hold with $\rho(y) = \exp(\|y\|)$ and $\gamma(t, y) = (1+\|y\|)$. Note **(A2.2)** and **(A2.4)** hold, and **(A2.5)** holds with $f^\dagger(x, y) = \gamma\|y\|\|x\|$ for some constant $\gamma$. The initial conditions were $x_1(0) = 2$, $x_2(0) = -2$, $y_1(0) = 0.75$, $y_2(0) = -0.75$ and $k(0) = 1$.

In Figure $(5-4)$, we see the evolution of $k(\cdot)$, showing the usual characteristics we expect in

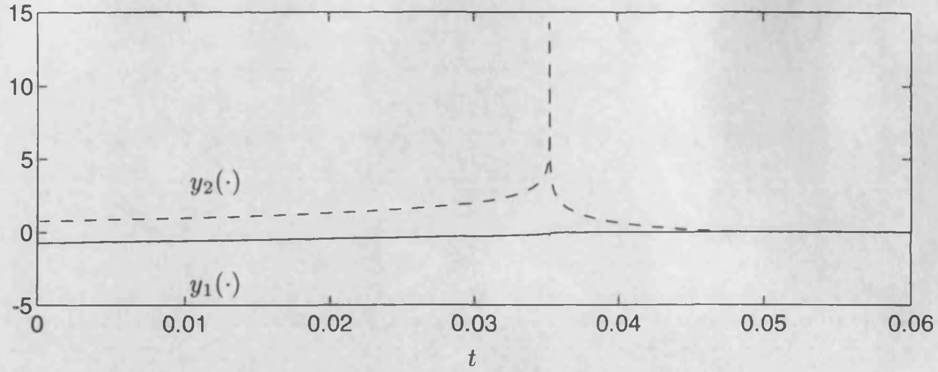


Figure 5-5: The evolution of $y(\cdot)$.

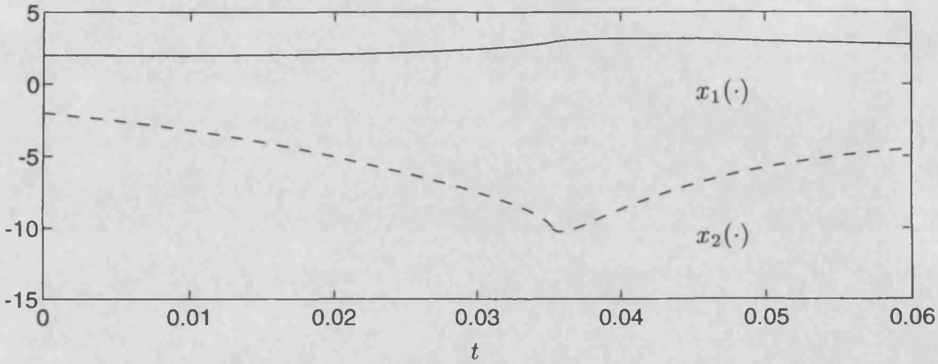


Figure 5-6: The evolution of $x(\cdot)$.

this form of control. $k(t)$ tends to an appropriate level, and then levels off.

Figure $(5-5)$ shows the evolution of the output. As expected by the theory this tends to zero, letting the zero dynamics take the state $x$ to zero as demonstrated in the Figure $(5-6)$. Here we see the state $x$ is tending to the origin, but slowly at a rate governed by the zero dynamics.

## 5.7 Exponentially-asymptotically stable zero-dynamics

The final subclass $\mathcal{C}_3$ of the class $\mathcal{C}$ that makes the origin globally attractive by the universal control (5.10) is classified by structural assumptions, **(A3.1)** to **(A3.4)** below. The subclass $\mathcal{C}_3$ of $\mathcal{C}$ is classified by the following:

**Structural Assumptions:**

**(A3.1):** The general assumptions **(G1)** to **(G4)** hold.

**(A3.2):** The function $x \mapsto \bar{f}(x)$ satisfies the following properties:

- $\bar{f}(x) = 0 \iff x = 0$,
- $x \mapsto \bar{f}(x)$ is globally Lipschitz.

- There exists a $c > 0$ and a $K > 0$ such that for $x^0 \in \mathbb{R}^n$, the solution $x(t)$ to $\dot{x}(t) = \bar{f}(x(t))$ with $x(t_0) = x^0$ satisfies

$$\|x(t)\| \leq K \exp(-c(t-t_0))\|x^0\|, \qquad \forall t \geq t_0,$$

ie. the origin is exponentially stable.

For some known continuous function $\rho : \mathbb{R}^m \to \bar{\mathbb{R}}_+$ satisfying, for all $r \in [0, \infty)$, $\|y\|^r \leq \gamma_{5,r}\rho(y)$ for all $y \in \mathbb{R}^m$, where $\gamma_{5,r}$ is a constant that can depend on $r$, the following hold.

**(A3.3):** The residue function, $f^*(\cdot,\cdot,\cdot)$, has the following bound:

$$\|f^*(t,x,y)\| \leq \gamma_1(1 + \|x\|^{\delta_1})\|y\|h^*(y), \qquad \forall (t,x,y) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m,$$

where $\gamma_1 > 0$ is a possibly unknown constant and $0 \leq \delta_1 < 1$. $h^* : \mathbb{R}^m \to \bar{\mathbb{R}}_+$ is continuous and satisfies the following: for all $r$ such that $1 \leq r < \infty$,

$$(h^*(y))^r \leq \gamma_{2,r}\rho(y), \qquad \forall y \in \mathbb{R}^m,$$

where $\gamma_{2,r}$ is a possibly unknown constant that can depend on $r$.

**(A3.4):** The function $g(\cdot,\cdot,\cdot)$ is in the class of functions such that

$$\|g(t,x,y)\| \leq \gamma_3(1 + \|x\|^{\delta_2})l(y), \qquad \forall (t,x,y) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m,$$

where $\gamma_3 > 0$ is a possibly unknown constant and $0 \leq \delta_2 < 2$. The function $l : \mathbb{R}^m \to \bar{\mathbb{R}}_+$ satisfies for all $r$ such that $1 \leq r < \infty$

$$(l(y))^r \leq \gamma_{4,r}\rho(y), \qquad \forall y \in \mathbb{R}^m,$$

where $\gamma_{4,r}$ is a possibly unknown constant that can depend on $r$.

See Chapter 4 for possible examples of functions that satisfy these conditions.

Again we make use of the exponential decay of the Liapunov function $V(\cdot)$ along the solution of the zero dynamics, obtained from the inverse Liapunov theory [83].

From the theory established in the single-input, single-output case, we are now able to give the main theorem of this section, detailing the stability of the system (5.7) under the control (5.10). It establishes (i) the maximal interval of existence of any solution to the differential inclusion (5.11) is the infinite interval $[t_0, \infty)$, (ii) the adaptive gain $k(\cdot)$ tends to a finite limit and (iii) the origin is globally attractive with respect to the solutions $x(t)$ and $y(t)$.

**Theorem 5.7.1** *Let $w : [t_0, \omega) \to \mathbb{R}^{n+m+1}$ be a solution to the differential inclusion (5.11) on its maximal interval of existence, then*

*(i)* $\omega = \infty$,

*(ii)* $\lim_{t\to\infty} k(t)$ *exists and is finite,*

*(iii)* $(x(t), y(t)) \to (0,0)$ *as* $t \to \infty$.

**Proof:** Assumption **(A3.2)** gives the existence of a Liapunov function $V(\cdot)$ that satisfies the properties of Theorem $B$.1.1 in the appendix. By analogous arguments, and the same $\zeta(t)$ as in Theorem 4.6.3 we get

$$V^{+}(x(t); \dot{x}(t)) + \frac{1}{2} qcV(x(t)) \leq \zeta(t). \tag{5.17}$$

To show the boundedness of $k(\cdot)$, consider the function $Y : \mathbb{R}^m \to \bar{\mathbb{R}}_+$, $Y : y \mapsto \|y\|^2/2$. For all $(t, w) \in \mathbb{R} \times \mathbb{R}^{n+m+1}$, $y \neq 0$,

$$\begin{aligned}
\langle y, \eta \rangle &\leq \langle y, g(t, x, y) \rangle + \max_{u \in \Psi(y,k)} \{ \langle y, H(t, x, y) u \rangle \} \\
&\leq \gamma_3 (1 + \|x\|^{\delta_2}) \|y\| l(y) - k\rho(y) \|y\|^{-1} \langle y, H(t, x, y) K(k) y \rangle \\
&\leq \epsilon^{-1} e_1 \|y\| \rho(y) \gamma(t, y) + V^2(x) - k\nu(k) \|y\| \rho(y) \gamma(t, y)
\end{aligned}$$

for all $\eta \in F_2(t, w)$, with $\nu(\cdot)$ as in (5.12) and constant $e_1$ as in Theorem 4.6.3. Therefore

$$\frac{dY(y(t))}{dt} \leq \epsilon^{-1} e_1 \|y(t)\| \rho(y(t)) \gamma(t, y(t)) + V^2(x(t)) - k(t)\nu(k(t)) \|y(t)\| \rho(y(t)) \gamma(t, y(t)).$$

So recalling $\dot{k}(t) = \|y(t)\| \rho(y(t)) \gamma(t, y(t))$,

$$0 \leq Y(y(t)) \leq Y(y(t_0)) + \epsilon^{-1} e_1 (k(t) - k(t_0)) + a_1 V^2(x(t_0)) + a_2 \int_{t_0}^{t} \zeta^2(s)\, ds - \int_{k(t_0)}^{k(t)} k\nu(k)\, dk.$$

by Lemma 4.6.2. This gives

$$0 \leq Y(y(t)) \leq Y(y(t_0)) + a_1 V^2(x(t_0)) + (e_1 \epsilon^{-1} + e_2)(k(t) - k(t_0)) - \int_{k(t_0)}^{k(t)} k\nu(k)\, dk. \tag{5.18}$$

with constant $e_2$ as in Theorem 4.6.3. By an analogous argument as given in Theorem 5.5.2, if $k(\cdot)$ is assumed to be unbounded, a contradiction arises. Therefore $k(\cdot)$ is bounded. This and (5.18) gives the boundedness of $y(\cdot)$.

Now we show that $x(\cdot)$ and therefore $w(\cdot)$ is bounded. From equation (5.18), the fact that $\zeta(t)$ is bounded by a constant multiple of $\|y(t)\| \rho(y(t))$, i.e.

$$\zeta(t) \leq (L\gamma_1 \gamma_{2,1} + c_2 (\gamma_{5,2(r_1-1)} \gamma_{2,2r_1})^{\frac{1}{2}}) \|y(t)\| \rho(y(t)),$$

$\rho(\cdot)$ being continuous and $y(\cdot)$ being bounded, we get

$$V^+(x(t);\xi) \leq -\frac{1}{2}qcV(x(t)) + K$$

where $K$ is some constant. A similar argument to that in Theorem 4.6.3 gives the boundedness of $x(\cdot)$ and in turn $w(\cdot)$. This implies $\omega = \infty$ by Theorem $A$.3.3 in the appendix.

All that remains to prove is $(x(t), y(t)) \to (0,0)$ as $t \to \infty$. By an analogous argument as that in Theorem 4.6.3, we get this result. □

### 5.7.1 Example

We show the universal control stabilises the following system, where $n = 2$ and $m = 3$:

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix} -1 & -25 \\ 25 & -1 \end{bmatrix} x(t) + 5\sin(t)\|y(t)\|^2 \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \\ \dot{y}(t) &= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T + y(t) + u\end{aligned}$$

All the controller need know about this system is that it belongs to the class of systems that satisfies the structural assumptions with $\rho(y) = \exp(\|y\|)$ and $\gamma(t,y) = 1$. Note $\delta_1 = 0$ and $\delta_2 = 0$. For this specific example we let $x(0) = [1 \ -1]$, $y(0) = [0.5 \ -0.25 \ 0]$ and $k(0) = 2$, obtaining the following results:

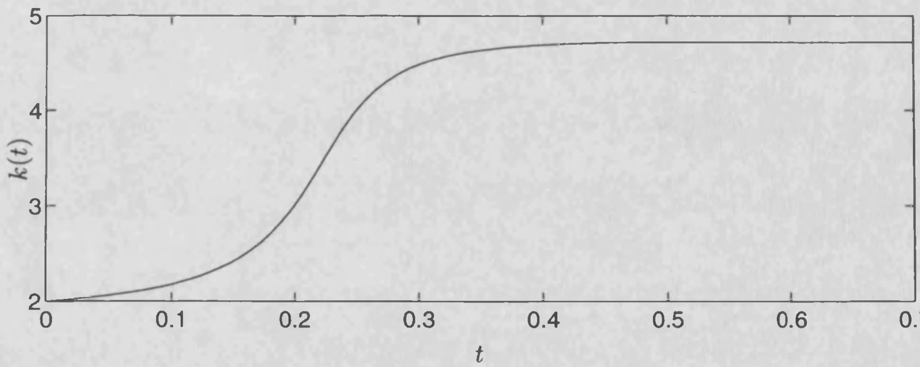


Figure 5-7: The evolution of $k(\cdot)$.

In Figure $(5-7)$ we see the familiar characteristics of the evolution of $k(\cdot)$, leveling out around $t = 0.5$. In Figure $(5-8)$ we see this is the time at which the output reaches the origin.

Figure $(5-9)$ shows the evolution of the state. The oscillations are increasing in magnitude to begin with, but as the output is force to zero around the time $t = 0.5$, we see the oscillations decrease in magnitude.

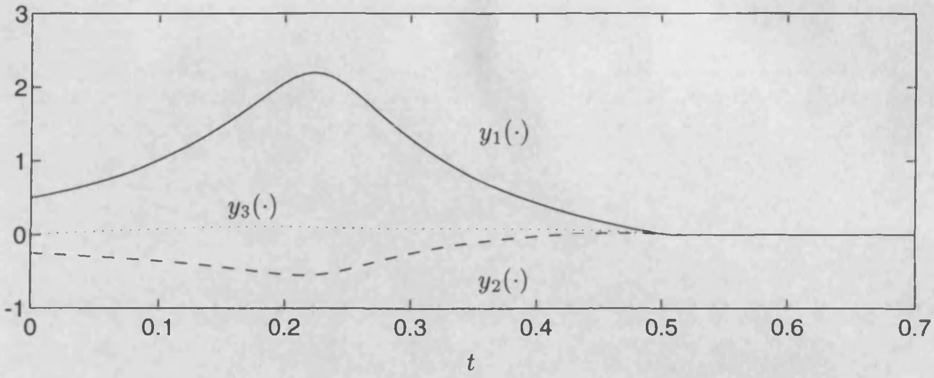


Figure 5-8: The evolution of $y(\cdot)$.

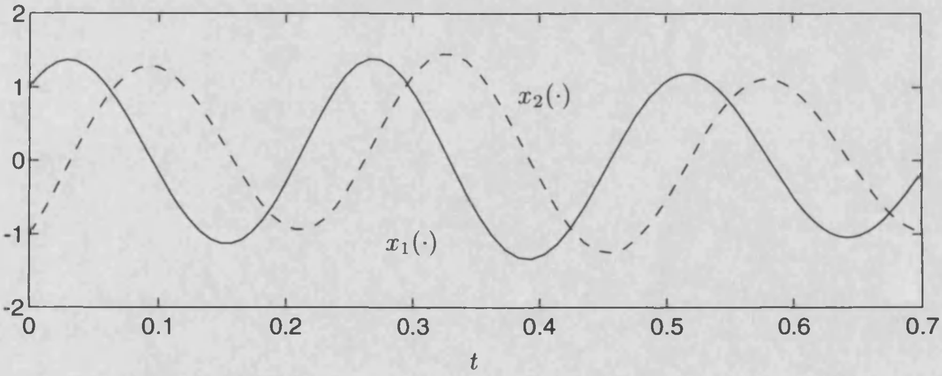


Figure 5-9: The evolution of $x(\cdot)$.

# Chapter 6

# Adaptive tracking of nonlinearly perturbed linear systems

## 6.1 Introduction

The control objective of the previous chapters has been to render the origin globally attractive under the feedback control. In this and the subsequent chapter, different control objectives are considered. Given a reference signal $r(\cdot)$ from a prescribed class, $\mathcal{R}$ say, the new objective is force some output of the system to 'track' this signal. There are two forms of tracking considered in this thesis. Firstly *asymptotic tracking* where the error between the output and the reference signal tends asymptotically to zero. Secondly $\lambda$-tracking where the error between the output and the reference signal tends asymptotically to the ball $\lambda\mathbb{B}$ of prescribed radius $\lambda > 0$.

The reader should note an important aspect of the latter control objective. So far, the controls of this thesis have had a variable structure, taking on two different forms on different sides of a 'switching surface' $\{(t,x,y) \mid y = 0\}$, which creates a discontinuity in the control at the boundary of this set. This may be undesirable, as this type of control can lead to 'chattering'. When this occurs the control switches between its two forms at a very high frequency. (This is unwanted in chemical systems, for example.) With the weaker control objective of $\lambda$-tracking, the discontinuity in the control can be removed. With this advantage in mind, a $\lambda$-neighbourhood of the origin can be made globally attractive, by using the $\lambda$-tracking feedback control by equating the signal to be tracked to be $r(t) = 0$ for all $t$. There is another advantage to using $\lambda$-tracking. There may be situations where the output is corrupted by noise $n(\cdot) : \mathbb{R} \to \mathbb{R}^m$. The $\lambda$-tracking feedback control can cope with noise such that $n(\cdot) \in W^{1,\infty}(\mathbb{R}, \mathbb{R}^m)$, by the following argument: if the output is supposed to track the reference signal $r(\cdot)$, the actual objective would be for the output to $\lambda$-track $r(\cdot) - n(\cdot)$, where $\lambda > \|n(\cdot)\|_{1,\infty}$.

The class $\mathcal{R}$ of reference signals permitted in this thesis is of special note. The members are signals $r : \mathbb{R} \to \mathbb{R}^m$, which are absolutely continuous on compact intervals and which are bounded with essentially bounded derivatives, ie. $r, \dot{r} \in L^\infty(\mathbb{R}; \mathbb{R}^m)$, see [71], [69], [72] and [38] (see also [30]). This function space is equipped with the norm

$$\|r\|_{1,\infty} = \|r\|_\infty + \|\dot{r}\|_\infty \tag{6.1}$$

where $\|\cdot\|_\infty$ is the $L^\infty$-norm. We note that this class of functions is the Sobolev space $W^{1,\infty}(\mathbb{R}, \mathbb{R}^m)$. This class of reference signals is large, when compared to other classes that are usually associated with tracking. Other work has restricted the reference signals to those that can be interpreted, either explicitly (as in model reference adaptive control [44]) or implicitly (as in, for example [55] and [77]) as solutions from differential equations. The class $\mathcal{R}$ is more general in nature, the members of which are, in general, not related to solutions of differential equations.

The specific class of systems to be studied are generally linear systems with nonlinear perturbations, as found in Chapters 2 and 3. Specifically

$$M^* z^{(p)}(t) + B^* u(t) = g(t, z(t), \dot{z}(t), \ldots, z^{(p-1)}(t)), \qquad z(t), u(t) \in \mathbb{R}^m \tag{6.2}$$

where $M^*$, $B^*$ and $g$ (assumed measurable in $t$ and continuous in $(z, \ldots, z^{(p-1)})$) are uncertain.

The two separate approaches to the multi-dimensional, linear based systems, found in Chapters 2 and 3, will be considered in this tracking scenario. We generalise the form of the controls of those chapters, to cater for the new control objective. However, the $\lambda$-tracking objective will not be solved using a control strategy similar to that of Chapter 2. This is due to the analytical problems that arise in the objective of $\lambda$-tracking for multi-input systems, see [38] for these problems. (In this latter reference the following, reasonably strong, structural assumptions are imposed on the high-frequency gain matrix. Its eigenvalues are wholly in the left or right hand complex plane, *and* knowledge of a positive definite, symmetric matrix $P$ is also required, such that $PM^{*-1}B^* + (M^{*-1}B^*)^T P$ is sign definite. This $P$ is needed in the control. By imposing this eigenvalue assumption on the high-frequency gain matrix, the demand for the knowledge of a viable $P$ is reasonable. However, in this thesis this eigenvalue assumption is not imposed and this could lead to impractical assumptions. For example, a possible assumption could require the knowledge of a matrix $P$ such that $PM^{*-1}B^*U + (M^{*-1}B^*U)^T P = I$. In this author's opinion, this structural assumption is too impractical.) We find, when compared to the assumptions imposed in Chapter 3, modifications are also needed in the general $m$-input case. Now the high-frequency gain matrix will have the property of row and column diagonal dominance in the $\alpha$, $\beta$ sense. The results of Chapter 5 will help give rise to a practical result for multi-input systems in the $\lambda$-tracking case.

The chapter is concluded with a brief look at the case where the full state is not available for feedback.

## 6.2 The servomechanism

The classes of system studied in this chapter are of the same general form as the systems encountered in Chapters 2 and 3. We study $m$-input linear based systems, with nonlinear perturbations, for general $m$. For the control objective of asymptotic tracking, the high-frequency gain matrix will just be invertible. When $m \geq 3$ the extra condition of column diagonal dominance will be imposed on this matrix. For the control objective of $\lambda$-tracking we impose the stricter condition of row and column diagonal dominance in the $\alpha$, $\beta$ sense on the constant high-frequency gain matrix. Explicitly the general class of system to be studied in this chapter will be the following:

$$\begin{aligned} &M^* z^{(p)}(t) + B^* u(t) = g(t, z(t), \dot{z}(t), \ldots, z^{(p-1)}(t)), \\ &(z(0), \dot{z}(0), \ldots, z^{(p-1)}(0)) \in \mathbb{R}^m \times \cdots \times \mathbb{R}^m \end{aligned} \tag{6.3}$$

where $z(t)$, $u(t) \in \mathbb{R}^m$. $M^*$, $B^*$ and $g$ (assumed measurable in $t$ and continuous in $(z, \dot{z}, \ldots, z^{(p-1)})$) are uncertain. The following structural assumptions are imposed:

**Structural Assumptions:**

**A1:** The objective and dimension dependent conditions shall be imposed on the high-frequency gain matrix $M^{*-1}B^* \in \mathcal{GL}(m; \mathbb{R})$:

- In the asymptotic tracking case for $m \geq 3$ we impose the extra condition that the matrix $M^{*-1}B^*$ is essentially column diagonally dominant (see Chapter 3 for this definition).
- In the $\lambda$-tracking objective case for general $m$ we impose the condition that the constant matrix $M^{*-1}B^*$ is essentially row and column diagonally dominant in the $\alpha$, $\beta$ sense.

**A2:** $g$ is bounded modulo an unknown scalar multiplier $\mu > 0$, by a known continuous function of the state, $\gamma$, in the sense that, for almost all $t \in \mathbb{R}$

$$\|g(t, v_1, v_2, \cdots, v_p)\| \leq \mu\gamma(v_1, \cdots, v_p), \qquad \forall\, (v_1, \cdots, v_p) \in \mathbb{R}^m \times \cdots \times \mathbb{R}^m.$$

An example of a typical $g$ can be found in the Chapter 2.

Similar to Chapters 2 and 3, write $M = \mu^{-1}M^*$ and $B = \mu^{-1}B^*$, and define the set-valued map

$$\mathcal{Z} : (v_1, \cdots, v_p) \mapsto \gamma(v_1, \cdots, v_p)\overline{\mathbb{B}}.$$

From the argument given in the Chapter 2, $\mathcal{Z}$ is a known, continuous set-valued map from $\mathbb{R}^{mp}$ to the non-empty, convex and compact subsets of $\mathbb{R}^m$. The system (6.3) can now be embedded in the differential inclusion:

$$\begin{aligned} &Mz^{(p)}(t) + Bu(t) \in \mathcal{Z}(z(t), \dot{z}(t), \ldots, z^{(p-1)}(t)), \\ &(z(t_0), \dot{z}(t_0), \ldots, z^{(p-1)}(t_0)) \in \mathbb{R}^m \times \cdots \times \mathbb{R}^m \end{aligned} \tag{6.4}$$

where $z(t)$, $u(t) \in \mathbb{R}^m$.

### 6.2.1 A co-ordinate transformation

We proceed as before with the following co-ordinate transformation. Let $C_i \in \mathbb{R}^{m \times m}$, $i = 1, 2, \cdots, p-1$ be such that all the eigenvalues of the linear system

$$S : z^{(p-1)}(t) + C_{p-1} z^{(p-2)}(t) + \cdots + C_2 \dot{z}(t) + C_1 z(t) = 0$$

lie in the open left half plane $\mathbb{C}_-$. Define the transformation $T$ as

$$T : (z(t), \dot{z}(t), \cdots, z^{(p-2)}(t)) \mapsto (w(t), y(t)),$$

where $w(\cdot)$ and $y(\cdot)$ are defined as in (2.4) and (2.5). This transformation takes (6.4) into the form

$$\begin{aligned} &\dot{w}(t) = L_1 w(t) + L_2 y(t) \\ &M[\dot{y}(t) + L_3 w(t) - C_{p-1} y(t)] + Bu(t) \in \mathcal{F}(w(t), y(t)) \\ &(w(t_0), y(t_0)) \in \mathbb{R}^{m(p-1)} \times \mathbb{R}^m \end{aligned} \tag{6.5}$$

where $\mathcal{F} := \mathcal{Z} o T^{-1}$ and $L_1$, $L_2$ and $L_3$ are linear. Again note that $\sigma(L_1) \subset \mathbb{C}_-$.

## 6.3 The controls for tracking objectives

We shall use the ideas behind the two different control strategies of Chapters 2 and 3 for the control objective of asymptotically tracking a given reference signal from the class of reference signals $W^{1,\infty}(\mathbb{R}, \mathbb{R}^m)$. We also give a feedback control for the $\lambda$-tracking objective in the general $m$-input case.

In all of the controls for the tracking objectives, the *error* between $y(\cdot)$ and the given reference signal $r(\cdot) \in W^{1,\infty}(\mathbb{R}, \mathbb{R}^m)$, is required.

**Notation:** The error between the state and the reference signal is defined as the function

$$t \mapsto e(t) := y(t) - r(t). \tag{6.6}$$

### 6.3.1 The adaptive feedback strategy for asymptotic tracking in the $m$-input case

For the control objective of asymptotic tracking, we consider the two feedback approaches given in Chapters 2 and 3.

#### $m = 2$ : asymptotic tracking with a control similar to that of Chapter 2

In Chapter 2 the strategy involved an alternating mechanism between two control structures. Each structure achieved the control objective of stabilising the system for one of the two distinct cases $|M^{-1}B| > 0$ or $|M^{-1}B| < 0$. Here this strategy is modified to yield a control that achieves the asymptotic tracking objective. Let $(\tau_n)_{n\in\mathbb{N}}$ be as in Chapter 2. Choose $\xi \in (0, \tau_1)$ sufficiently small so that $\tau_2 > \tau_1 + \xi$, and define the continuous function $s : \mathbb{R} \to [-1, 1]$ as in (2.13). Letting $x = (w, e, k)$, the control strategy for the objective of asymptotically tracking a reference signal is therefore given as

$$\begin{aligned} u(t) &\in \Psi_1(t, x(t)) := k^2(t)[f(w(t), e(t) + r(t)) + \|e(t)\| + 1]\mathcal{O}_+(k(t))\mathcal{S}(s(k(t)))\psi(e(t)) \\ \dot{k}(t) &= \Psi_1^*(t, x(t)) := [f(w(t), e(t) + r(t)) + \|e(t)\| + 1]\|e(t)\|, \qquad k(0) = k^0 \in \mathbb{R}_+ \end{aligned} \tag{6.7}$$

where $\psi(\cdot)$ is defined as in (2.7), and $f$ is the continuous map $f : (w, y) \mapsto \max\{\|\phi\| \mid \phi \in \mathcal{F}(w, y)\}$. Note the main difference in the control is the appearance of the error, and the extra 1.

#### Asymptotic tracking in the general $m$-input case

The control in this case is similar in nature to (6.7). As might be expected, the control for asymptotic tracking in the general $m$-input case will involve a strategy cycling between the elements of $\mathcal{I}_n^E$, ie. the control will involve $K(\cdot)$ as defined in (3.4). We also expect the error $e(\cdot)$ to appear in a similar manner as in (6.7), and this is indeed the case. The control is given as the following. Let $x = (w, e, k)$,

$$\begin{aligned} u(t) &\in \Psi_2(t, x(t)) := -k(t)[f(w(t), e(t) + r(t)) + \|e(t)\| + 1]K(k(t))\psi(e(t)) \\ \dot{k}(t) &= \Psi_2^*(t, x(t)) := [f(w(t), e(t) + r(t)) + \|e(t)\| + 1]\|e(t)\|, \qquad k(0) \in \mathbb{R}_+ \end{aligned} \tag{6.8}$$

where again $f : (w, y) \mapsto \max\{\|\phi\| \mid \phi \in \mathcal{F}(w, y)\}$ and $\psi(\cdot)$ is as defined in (2.7).

#### $\lambda$-tracking in the general $m$-input case

With the control objective of $\lambda$-tracking, $\lambda > 0$, we would expect the adaptive mechanics to be 'active' when the error is outside the specified allowable magnitude $\lambda$. This is in fact the idea behind the control, see references [72] and [38] for example. For this purpose, the following function is defined. Let $d_\lambda : \mathbb{R}^m \to \bar{\mathbb{R}}_+$ be the Euclidean distance function for the ball $\lambda\mathbb{B}$

$$d_\lambda(e) := \begin{cases} \|e\| - \lambda, & \|e\| > \lambda \\ 0, & \|e\| \leq \lambda \end{cases}. \tag{6.9}$$

This function is clearly Lipschitz continuous. Let $\psi_\lambda(\cdot) : \mathbb{R}^m \to \mathbb{B}_m$ be any continuous function satisfying the property:

- $d_\lambda(e) \geq 0 \implies \psi_\lambda(e) = \|e\|^{-1}e$.

An example, the one given in [38], is

$$\psi_\lambda(e) := \begin{cases} e/\|e\|, & \|e\| > \lambda \\ e/\lambda, & \|e\| \leq \lambda \end{cases}$$

Let $\mathcal{I}_m^E$ be the finite set of Chapter 3, $(\tau_n)$ the sequence of the same chapter and $K(\cdot)$ be the continuous function (3.4) which cycles through the elements of the finite unmixing set $\mathcal{I}_m^E$, dictated by the sequence $(\tau_n)$. With $x = (w, e, k)$, we define the control

$$\begin{aligned} u(t) &= \Psi_3(t, x(t)) := -k(t)[f(w(t), e(t) + r(t)) + \|e(t)\| + d_\lambda(e(t)) + 1]K(k(t))\psi_\lambda(e(t)) \\ \dot{k}(t) &= \Psi_3^*(t, x(t)) := d_\lambda(e(t))[f(w(t), e(t) + r(t)) + \|e(t)\| + d_\lambda(e(t)) + 1],\ k(0) \in \mathbb{R}_+ \end{aligned} \tag{6.10}$$

where again $f : (w, y) \mapsto \max\{\|\phi\| \mid \phi \in \mathcal{F}(w, y)\}$.

### 6.3.2 Embedding the feedback system into a differential inclusion

Using the controls of the previous subsection, the transformed system (6.5) can be embedded in the following differential inclusion. Let $x = (w, e, k) \in \mathbb{R}^{m(p-1)} \times \mathbb{R}^m \times \mathbb{R}$. Define the function $F : \mathbb{R}^{mp+1} \to 2^{\mathbb{R}^{mp+2}}$ as

$$F(t, x) := F_1(t, x) \times F_2(t, x) \times F_3(t, x) \tag{6.11}$$

where $F_i(\cdot, \cdot)$, $i \in \{1, 2, 3\}$, are defined by

$$\begin{aligned} &F_1(t, x) := \{L_1 w + L_2(e + r(t))\} \\ &F_2(t, x) := \\ &\begin{cases} \{M^{-1}[\phi - Bu] - L_3 w + C_{p-1}y(t) - \xi : \|\xi\| \leq \|r(\cdot)\|_{1,\infty},\ \phi \in \mathcal{F}(w, y(t)),\ u \in \Psi_i(t, x)\}, \\ \qquad i \in \{1, 2\} \\ \{M^{-1}[\phi - Bu] - L_3 w + C_{p-1}y(t) - \xi : \|\xi\| \leq \|r(\cdot)\|_{1,\infty},\ \phi \in \mathcal{F}(w, y(t)),\ u = \Psi_i(t, x)\}, \\ \qquad i = 3 \end{cases} \\ &F_3(t, x) := \{\Psi_i^*(x)\} \end{aligned} \tag{6.12}$$

for some $i \in \{1, 2, 3\}$ suitably chosen, depending on the dimension $m$ and the control objective, (see the previous subsections). ($i = 1$ when $m = 2$ and the objective is asymptotic tracking, $i = 2$ when $m \geq 3$ and the objective is asymptotic tracking and $i = 3$ when the objective is $\lambda$-tracking.) The function $\Psi_i^*(\cdot,\cdot)$ is defined as the function

$$\Psi_i^*(t,x) = \begin{cases} (f(w, e + r(t)) + \|e\| + 1)\|e\|, & i = 1, 2 \\ d_\lambda(e)(f(w, e + r(t)) + \|e\| + d_\lambda(e) + 1), & i = 3. \end{cases} \tag{6.13}$$

The system (6.5) can now be embedded in the initial value problem

$$\dot{x}(t) \in F(t, x(t)), \qquad x(t_0) = x^0. \tag{6.14}$$

Observe $F(\cdot,\cdot)$ is upper-semicontinuous with compact and convex, non-empty values, so for each $(t_0, x_0)$ a (non-unique) solution can be guaranteed on a maximal interval $[t_0, \omega)$ by Theorem $A$.3.1 and Theorem $A$.3.2. Let such a solution be $x : [t_0, \omega) \to \mathbb{R}^{mp+1}$.

## 6.4 Tracking analysis

In this section we provide the analysis which establishes the success of the feedback strategy in its control objective. We begin with the objective of asymptotic tracking, completing the section with $\lambda$-tracking. Much of the analysis in the proofs of this section will exploit theory already established in previous chapters.

The following will be proved: with $x : [t_0, \omega) \to \mathbb{R}^{mp+1}$ a solution to the initial value problem (6.14), on a maximal interval of existence, we establish (i) the adaptive gain tends to a finite limit, (ii) $w(\cdot)$ is bounded, (iii) the maximal interval of existence of any solution is $[t_0, \infty)$ and (iv) the error either asymptotically tends to the origin or the error asymptotically tends to the ball $\lambda\mathbb{B}$, depending on the control objective.

### 6.4.1 Asymptotic tracking analysis for the $m = 2$ case

Here we focus on the analysis where a control strategy similar to that of Chapter 2 is used.

**Theorem 6.4.1** *Let $x(\cdot) = (w(\cdot), e(\cdot), k(\cdot)) : [t_0, \omega) \to \mathbb{R}^{mp+1}$ be a maximal solution of (6.14), then*

*(i)* $\omega = \infty$,

*(ii)* $\lim_{t\to\infty} k(t)$ *exists, and is finite,*

*(iii)* $x(\cdot)$ *is bounded and*

*(iv)* $\lim_{t\to\infty} \|e(t)\| = 0$.

**Proof** Let $D, Q$ and $O$ be as in Theorem 2.2.2. Recall $O$ has the form $\mathcal{S}(|O|)\mathcal{O}_+(\beta)$, for some unknown $\beta \in [0, 2\pi)$.

Let $W_1$ be as in Theorem 2.2.2, and define $W_2$ in the following way:

$$W_2(e) := \langle e, Q^{-1}e \rangle.$$

Recall the definition and properties of the function $H(\cdot)$, see Theorem 2.2.2,

$$H(k) := \begin{cases} -k^2 \cos(k+\beta), & |O|s(k) = +1 \\ -k^2 \cos(k+\beta) + 2k^2, & |O|s(k) \in [-1, +1). \end{cases}$$

In particular, recall

$$\begin{aligned} -k^2 \left\langle e, Q^{-1} D\mathcal{O}_+(k)\mathcal{S}(s(k))e \right\rangle &= -k^2 \langle e, \mathcal{S}(|O|)\mathcal{O}_+(k+\beta)\mathcal{S}(s(k))e\rangle \\ &\le H(k)\|e\|^2, \qquad \forall (k, e) \in \mathbb{R} \times \mathbb{R}^m. \end{aligned}$$

Therefore, for all $\eta \in F_2(t, x)$, we have

$$\langle \nabla W_2(e), \eta \rangle \le -\langle Q^{-1}e, L_3 w \rangle + (c_2 + H(k))[f(w, y) + \|e\| + 1]\|e\|, \qquad \forall x = (w, e, k) \in \mathbb{R}^{mp+1},$$

where the constant $c_2$ is the same constant as that found in the proof of Theorem 2.2.1 but with the addition of $\|r\|_{1,\infty}(\|Q^{-1}C_{p-1}\| + \|Q^{-1}\|)$. We begin by showing the boundedness of $k(\cdot)$. It follows, for almost all $t \in [t_0, \omega)$,

$$\frac{d}{dt}(W_2(e(t))) \le -\langle Q^{-1}e(t), L_3 w(t) \rangle + (c_2 + H(k(t)))\dot{k}(t)$$

which gives

$$0 \le W_2(e(t)) \le a_1 + a_2(k(t) - k(t_0)) + \int_{k(t_0)}^{k(t)} H(k)\, dk, \tag{6.15}$$

where $a_1 = W_2(e(t_0)) + c_0\|Q^{-1}L_3\|\|w(t_0)\|^2$ and $a_2 = (c_1 + c_3)\|Q^{-1}L_3\| + c_2$, where $c_0$, $c_1$ and $c_3$ are constants from Lemma $D.1.1$ in the appendix. (Note $\rho = \|L_2\|\|r\|_{1,\infty}$ in Lemma $D.1.1$ and $\dot{k}(t) \ge \|e(t)\|(1 + \|e(t)\|)$.) By an analogous argument as that in Theorem 2.2.2, if $k(\cdot)$ is assumed to be bounded, a contradiction arises. Therefore $k(\cdot)$ must be bounded.

Now, the boundedness of $w(\cdot)$ and $e(\cdot)$ are considered, to show the solution of (6.14) exists on the infinite interval $[t_0, \infty)$. We get the boundedness of $e(\cdot)$ immediately from (6.15), since $k(\cdot)$ is bounded.

The boundedness of $w(\cdot)$ is established from the following. Recall $\|e(\cdot)\|_\infty \le c$ for some constant *c.* Since $\dot{w}(t) \in F_1(t, x(t)) = \{L_1 w(t) + L_2(e(t) + r(t))\}$ we have for all $t \in [t_0, \omega)$

$$\|w(t)\| \leq \|\exp(L_1(t-t_0))\| \|w(t_0)\| + \int_{t_0}^{t} \|\exp(L_1(t-t_0))\| [\|L_2\|(c + \|r\|_{1,\infty})] \, ds.$$

Since $\sigma(L_1) \subset \mathbb{C}_-$, $w(\cdot)$ is bounded. So by Theorem *A*.3.3 in the appendix $\omega = \infty$.

It remains to prove $e(t) \to 0$ as $t \to \infty$. This is simply done by the use of Lemma *D*.2.1 in the appendix. Since $\dot{k}(t) \geq \|e(t)\|^2$, $e(\cdot) \in L^2$. Also from the dynamics of $e(\cdot)$ in (6.14), $\dot{e}(\cdot) \in L^\infty$. (Recall $u \in \Psi_1(t, x)$, and for bounded $K$, $\Psi(\mathbb{R}, K)$ is bounded.) So by Lemma *D*.2.1 in the appendix, $e(t) \to 0$ as $t \to \infty$. □

### Example

We now give an example, $p = 2$ that belongs to the class of systems that satisfies the structural assumptions of this section with $\gamma(v_1, v_2) = \exp(\|v_1\| + \|v_2\|)$. This is example 1 of Chapter 2:

$$\ddot{z}(t) + \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} u = \begin{bmatrix} -z_1^{\frac{1}{3}}(t) + \cos(z_1(t)\dot{z}_1(t)) \\ |\dot{z}_2(t)|^{\frac{1}{2}} \end{bmatrix}.$$

For the control purposes, only the function $\gamma(\cdot, \cdot)$ need be known.

Recall the constructed output is $y(t) := z(t) + \dot{z}(t)$.

The function this output shall track shall be generated from the well known Lorenz system, see [78],

$$\left.\begin{aligned} \dot{r}_1(t) &= -10(r_1(t) - r_2(t)) \\ \dot{r}_2(t) &= 28 r_1(t) - r_2(t) - 30 r_1(t) r_3(t) \\ \dot{r}_3(t) &= 30 r_1(t) r_2(t) - \tfrac{8}{3} r_3(t) \\ r(0) &:= (r_1(0), r_2(0), r_3(0)) = (1, 1, 0) \end{aligned}\right\} \tag{6.16}$$

This system exhibits "chaotic" behaviour, as discussed by J. M. T. Thompson and H. B. Stewart in [78]. (Note the system given above is the Lorenz system given in [78] but with a scaling factor of 30 applied.) The evolution of this system can be found in Figures $(6-1)$, $(6-2)$ and $(6-3)$.

The function we track is $[r_1(t)\ r_2(t)]^T$. With initial conditions $z(0) = [0.2\ \ -0.2]$, $\dot{z}(0) = [-0.1\ 0.1]$ and $k(0) = 0.5$, the following results, see Figures $(6-4)$, $(6-5)$ and $(6-6)$, are obtained.

Again, in Figure $(6-4)$, the evolution of $k(\cdot)$ is as expected, levelling out as the constructed output tracks the function $r(\cdot)$, as seen in Figures $(6-5)$ and $(6-6)$.

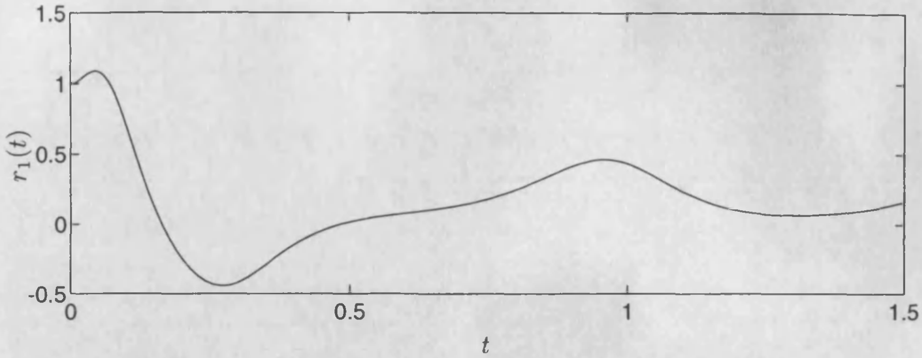


Figure 6-1: The evolution of $r_1(\cdot)$.

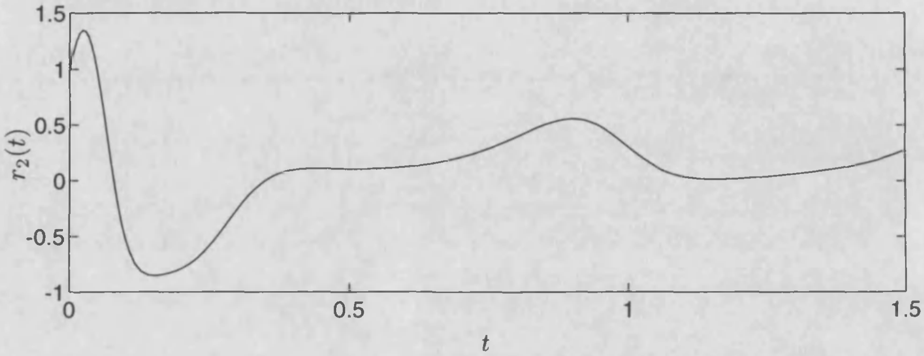


Figure 6-2: The evolution of $r_2(\cdot)$.

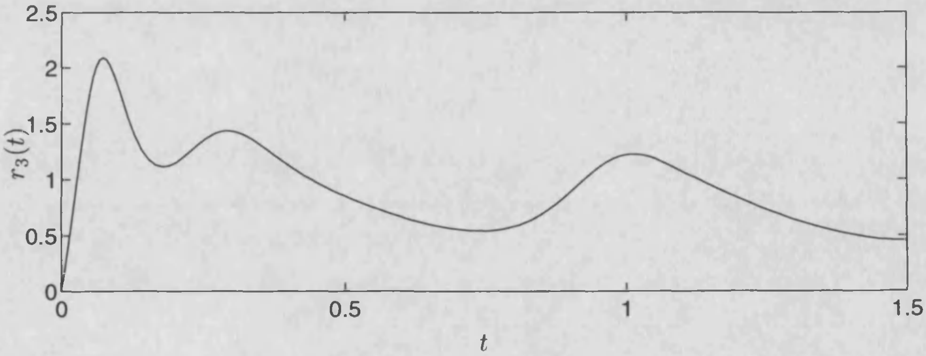


Figure 6-3: The evolution of $r_3(\cdot)$.

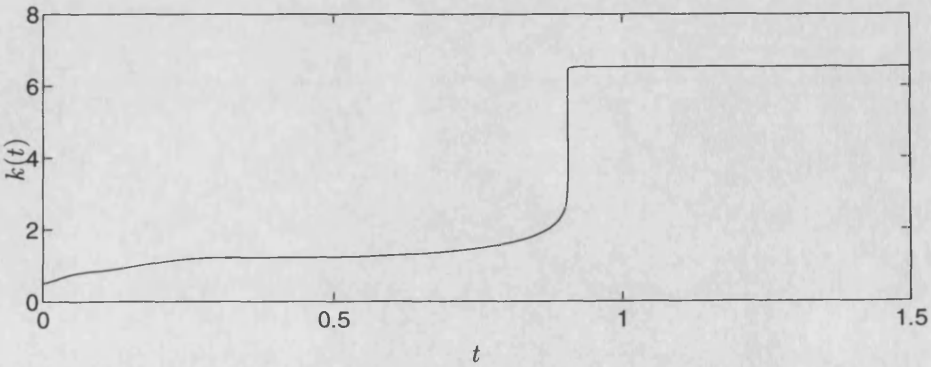


Figure 6-4: The evolution of $k(\cdot)$.

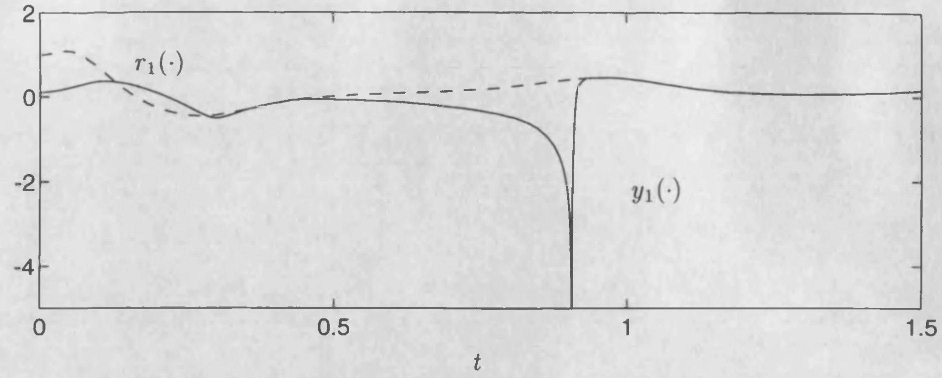


Figure 6-5: The evolution of $y_1(\cdot)$ and $r_1(\cdot)$.

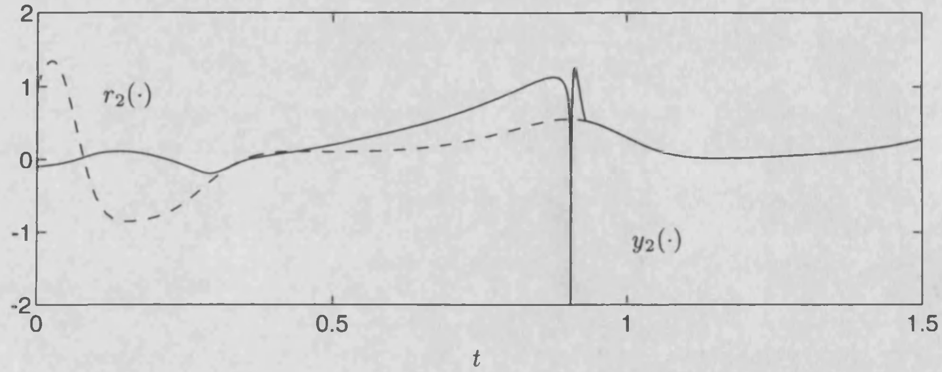


Figure 6-6: The evolution of $y_2(\cdot)$ and $r_2(\cdot)$.

### 6.4.2 Asymptotic tracking analysis for the general $m$-input case

Again the problem of asymptotic tracking is addressed in the following theorem where general $m$-inputs are permitted.

**Theorem 6.4.2** *Let $x = (w, e, k) : [t_0, \omega) \to \mathbb{R}^{mp+1}$ be a solution of the differential inclusion (6.14), on its maximal interval of existence, then*

*(i)* $\omega = \infty$,

*(ii)* $\lim_{t\to\infty} k(t)$ *exists and is finite,*

*(iii)* $x(\cdot)$ *is bounded,*

*(iv)* $\lim_{t\to\infty} \|e(t)\| = 0$.

**Proof:** Let $Q$ be the positive-definite symmetric matrix of Lemma 3.5.1 and $P$ be the positive definite symmetric matrix such that

$$PL_1 + (PL_1) = -I.$$

Define the functions

$$W_1 : w \mapsto \frac{1}{2}\langle w, Pw\rangle \text{ and } W_2 : y \mapsto \frac{1}{2}\langle e, Qe\rangle.$$

Since $\dot{w}(t) = L_1 w(t) + L_2(e(t) + r(t))$ with $\sigma(L_1) \subset \mathbb{C}_-$, from Lemma $D.1.1$ in the appendix there exists constants $c_0$, $c_1$ and $c_2$ such that for all $t \in [t_0, \omega)$,

$$\begin{aligned}\int_{t_0}^{t} \|e(s)\|\|w(s)\|\,ds &\leq c_0\|w(t_0)\|^2 + c_1\int_{t_0}^{t}\|e(s)\|\,ds + c_2\int_{t_0}^{t}\|e(s)\|^2\,ds \\ &\leq c_0\|w(t_0)\|^2 + (c_1 + c_2)(k(t) - k(t_0))\end{aligned}$$

using the dynamics of $k(\cdot)$ for the last inequality. For all $(t, x) \in \mathbb{R} \times \mathbb{R}^{mp+1}$,

$$\begin{aligned}\langle \nabla W_2(e), \eta\rangle &\leq \max_{\phi\in\mathcal{F}(w,y)}\{\langle Qe, M^{-1}\phi\rangle\} + \max_{u\in\Psi_2(t,x)}\{-\langle Qe, M^{-1}Bu\rangle\} \\ &\quad + \langle Qe, -L_3 w + C_{p-1}e\rangle + \|r\|_{1,\infty}(\|Q\| + \|QC_{p-1}\|)\|e\| \\ &\leq f(w,y)\|QM^{-1}\|\|e\| + \|QC_{p-1}\|\|e\|^2 + \|QL_3\|\|e\|\|w\| \\ &\quad + \|r\|_{1,\infty}(\|QC_{p-1}\| + \|Q\|)\|e\| + \max_{u\in\Psi_2(t,x)}\{-\langle Qe, M^{-1}Bu\rangle\}.\end{aligned}$$

for all $\eta \in F_2(t, x)$. Since $M^{-1}B$ is essentially column diagonally dominant and invertible, Lemma 3.5.1 gives for $u \in \Psi_2(t, x)$,

$$-\langle Qe, M^{-1}Bu\rangle \leq -k[f(w, e + r(t)) + \|e\| + 1]\|e\|\nu(k), \qquad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^{mp+1}$$

where $\nu(\cdot)$ is defined as in Lemma 3.5.2. Writing $c_3 = \|QC_{p-1}\| + \|QM^{-1}\| + \|r\|_{1,\infty}(\|QC_{p-1}\| + \|Q\|)$, this gives for all $\eta \in F_2(x)$,

$$\langle \nabla W_2(e), \eta\rangle \leq \|QL_3\|\|e\|\|w\| + (c_3 - k\nu(k))[f(w, y) + \|e\| + 1]\|e\|, \forall (t, x) \in \mathbb{R} \times \mathbb{R}^{mp+1}.$$

and consequently

$$\frac{dW_2(e(t))}{dt} \leq \|QL_3\|\|e(t)\|\|w(t)\| + (c_3 - k(t)\nu(k(t)))[f(w(t), y(t)) + \|e(t)\| + 1]\|e(t)\|,$$

for almost all $t \in [t_0, \omega)$. Therefore by Lemma $D.1.1$ in the appendix, for all $t \in [t_0, \omega)$

$$\begin{aligned}0 \leq W_2(e(t)) &\leq W_2(e(t_0)) + c_0\|QL_3\|\|w(t_0)\|^2 \\ &\quad + ([c_1 + c_2]\|QL_3\| + c_3)(k(t) - k(t_0)) - \int_{k(t_0)}^{k(t)} \theta\nu(\theta)\,d\theta.\end{aligned} \tag{6.17}$$

By assuming $k(\cdot)$ is unbounded and using an analogous argument as given in Theorem 3.5.3, using Lemma 3.5.2, a contradiction arises. Therefore $k(\cdot)$ is bounded. From (6.17) and the boundedness of $k(\cdot)$, the boundedness of $e(\cdot)$ is established.

The boundedness of $w(\cdot)$ is established by an analogous argument as given in Theorem 6.4.1. Therefore $x(\cdot) = (w(\cdot), e(\cdot), k(\cdot))$ is bounded, implying $\omega = \infty$ by Theorem $A.3.3$ in the appendix.

All that remains to be proved is $e(t) \to 0$ as $t \to \infty$. This is done in an analogous way as that in Theorem 6.4.1. □

**Example**

With the same example of the last section, the same reference signal $[r_1(\cdot)\ r_2(\cdot)]^T$ with the same initial conditions and with $z(0) = [0.75\ -0.25]^T$, $\dot{z}(0) = [-0.5\ -0.25]^T$ and $k(0) = 1$, we obtain the results, see Figures $(6-7)$, $(6-8)$ and $(6-9)$, using this approach to the feedback control:

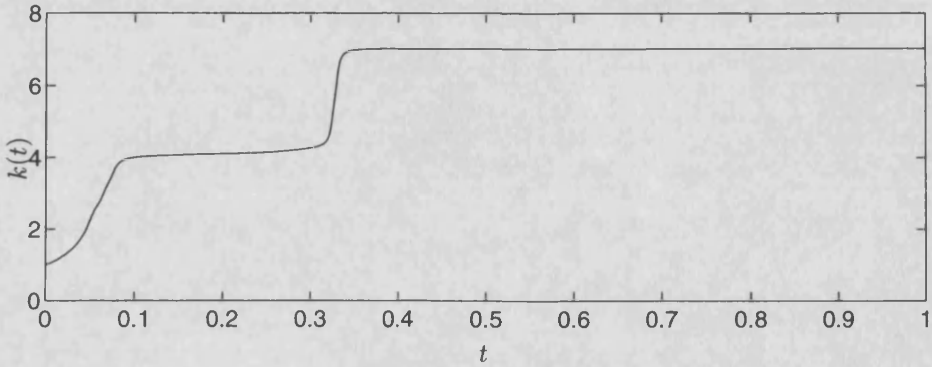


Figure 6-7: The evolution of $k(\cdot)$.

In Figure $(6-7)$, the evolution of $k(\cdot)$ is as expected, and the output tracks the function $r(\cdot)$

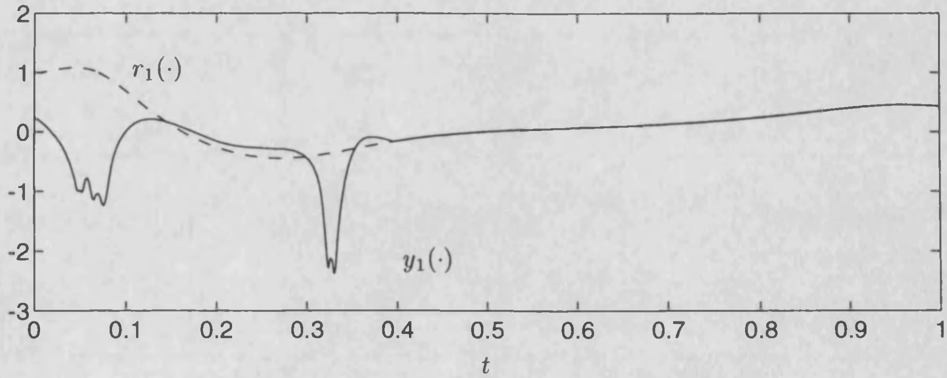


Figure 6-8: The evolution of $y_1(\cdot)$ and $r_1(\cdot)$.

as seen in Figure $(6-8)$ and Figure $(6-9)$.

### 6.4.3 Analysis of $\lambda$-tracking for the general $m$-input case

We now move to the analysis involved in the $\lambda$-tracking objective. The only notable difference in this proof is to show the norm of the error asymptotically tends to the ball $\lambda\mathbb{B}$.

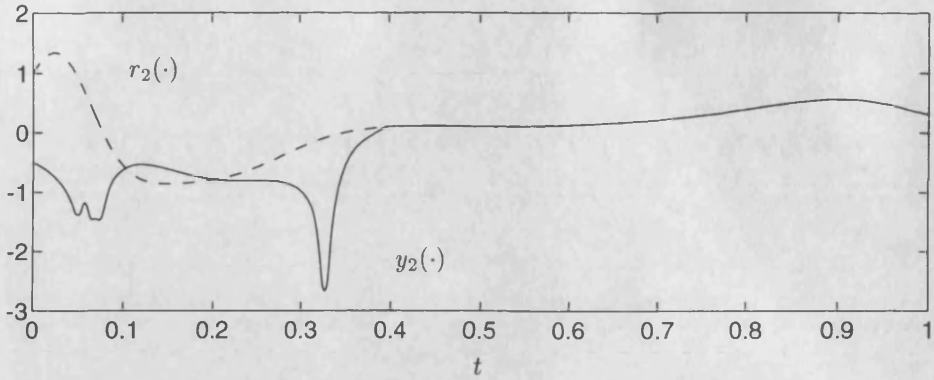


Figure 6-9: The evolution of $y_2(\cdot)$ and $r_2(\cdot)$.

**Theorem 6.4.3** *Let $x = (w, e, k) : [t_0, \omega) \to \mathbb{R}^{mp+1}$ be a solution of the differential inclusion (6.14), on the maximal interval of existence, then*

*(i)* $\omega = \infty$,

*(ii)* $\lim_{t\to\infty} k(t)$ *exists and is finite,*

*(iii)* $x(\cdot)$ *is bounded,*

*(iv)* $\lim_{t\to\infty} d_\lambda(e(t)) \to 0$.

**Proof:** Let $P$ be the matrix of Theorem 6.4.2, and define $W_1(\cdot)$ as in Theorem 6.4.2. Recall the definition of $d_\lambda : \mathbb{R}^m \to \bar{\mathbb{R}}_+$ in (6.9). From this define the $C^1$ function $W_2 : \mathbb{R}^m \to \bar{\mathbb{R}}_+$ as

$$W_2(e) := \frac{1}{2} d_\lambda^2(e).$$

In fact $\nabla W_2(e) = d_\lambda(e)\|e\|^{-1}e$ for $\|e\| \notin [0, \lambda]$ and $\nabla W_2(e) = 0$ for $\|e\| \in [0, \lambda]$. Since $\dot{w}(t) = L_1 w(t) + L_2 e(t) + L_2 r(t)$ with $\sigma(L_1) \subset \mathbb{C}_-$, the inequality

$$\int_{t_0}^{t} d_\lambda(e(s))\|w(s)\|\, ds \le \mu \int_{t_0}^{t} d_\lambda(e(s))[1 + d_\lambda(e(s))]\, ds \tag{6.18}$$

holds by Lemma $D$.1.1 in the appendix (setting $\theta(s) = \nabla W_2(e(s))$ in this lemma and noting $\|e - \nabla W_2(e)\| \le \lambda$). For all $(t, x) \in \mathbb{R} \times \mathbb{R}^{mp+1}$, $e \neq 0$,

$$\begin{aligned}
\langle \nabla W_2(e), \eta\rangle \le{}& d_\lambda(e)\|e\|^{-1}[\max_{\phi\in\mathcal{F}(w,y)}\{\langle e, M^{-1}\phi\rangle\} - \langle e, M^{-1}Bu\rangle \\
&+ \langle e, -L_3 w + C_{p-1}e + C_{p-1}r\rangle + \max_{\|\xi\|\le\|r\|_{1,\infty}}\{-\langle e, \xi\rangle\}] \\
\le{}& \|M^{-1}\|f(w,y)d_\lambda(e) + \|C_{p-1}\|d_\lambda(e)\|e\| + \|L_3\|d_\lambda(e)\|w\| \\
&+ \|r\|_{1,\infty}(\|C_{p-1}\| + 1)d_\lambda(e) - d_\lambda(e)\|e\|^{-1}\langle e, M^{-1}Bu\rangle.
\end{aligned}$$

for all $\eta \in F_2(t,x)$. Since the constant matrix $M^{-1}B$ is essentially row and column diagonally dominant in the $\alpha$, $\beta$ sense, for some $\alpha$ and $\beta$ such that $0 < \alpha \leq \beta$, Lemma 5.5.1 gives

$$-d_\lambda(e)\langle e, M^{-1}BK(k)\psi_\lambda(e)\rangle \leq -\nu(k)d_\lambda(e)\|e\|$$

where $\nu(\cdot)$ is defined as in Lemma 3.5.2 and by noting the function $d_\lambda(\cdot)$ compensates for $\psi_\lambda(\cdot)$ when $\|e\| < \lambda$. Writing $c_3 := \|M^{-1}\|+\|C_{p-1}\|+\|r\|_{1,\infty}(\|C_{p-1}\|+1)$, this gives for all $\eta \in F_2(t,x)$,

$$\langle \nabla W_2(e), \eta\rangle \leq \|L_3\|d_\lambda(e)\|w\| + (c_3 - k\nu(k))d_\lambda(e)[f(w,y) + d_\lambda(e) + \|e\| + 1],$$

and consequently

$$\frac{dW_2(e(t))}{dt} \leq \|L_3\|d_\lambda(e(t))\|w(t)\| + (c_3 - k(t)\nu(k(t)))d_\lambda(e(t))[f(w(t),y(t)) + d_\lambda(e(t)) + \|e(t)\| + 1].$$

Therefore for all $t \in [t_0, \omega)$, and using (6.18),

$$0 \leq W_2(e(t)) \leq W_2(e(t_0)) + (\mu\|L_3\| + c_3)(k(t) - k(t_0)) - \int_{k(t_0)}^{k(t)} \theta\nu(\theta)\, d\theta. \qquad (6.19)$$

By assuming $k(\cdot)$ is unbounded and using an analogous argument as given in Theorem 2.2.2, using Lemma 3.5.2, a contradiction arises. Therefore $k(\cdot)$ is bounded. From (6.19) and the boundedness of $k(\cdot)$, the boundedness of $e(\cdot)$ is established.

An analogous argument as in Theorem 6.4.1 gives the boundedness of $w(\cdot)$. Therefore $x(\cdot) = (w(\cdot), e(\cdot), k(\cdot))$ is bounded, implying $\omega = \infty$ by Theorem $A$.3.3 in the appendix.

All that remains to be proved is $d_\lambda(e(t)) \to 0$ as $t \to \infty$. This is done with a similar argument as in Theorem 6.4.2. First note that $\dot{k}(t) \geq d_\lambda^2(e(t))$. Since $k(\cdot)$ is bounded, $d_\lambda^2(e(\cdot)) \in L_1$. Also from the dynamics of $e(\cdot)$ in (6.14) we establish $\frac{d}{dt}(d_\lambda^2(e(\cdot))) \in L_\infty$. From this and Lemma $D$.2.1 in the appendix, we obtain the result $d_\lambda^2(e(t)) \to 0$ as $t \to \infty$ as required. □

### Example

Note that we cannot exactly use the previous examples of this chapter to demonstrate this $\lambda$-tracking case. This is because the matrix

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

is not a suitable $M^{*-1}B^*$ in this case, as it is not row and column diagonally dominant in the

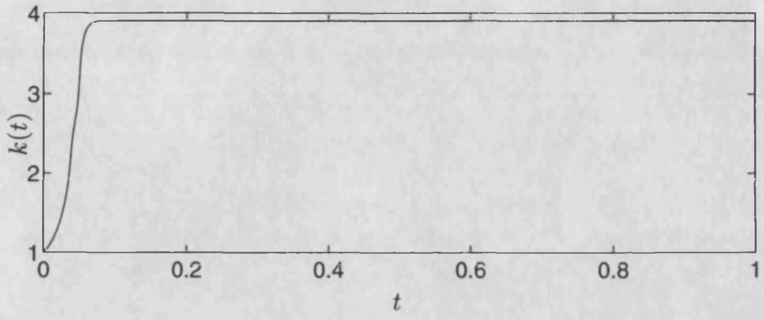


Figure 6-10: The evolution of $k(\cdot)$.

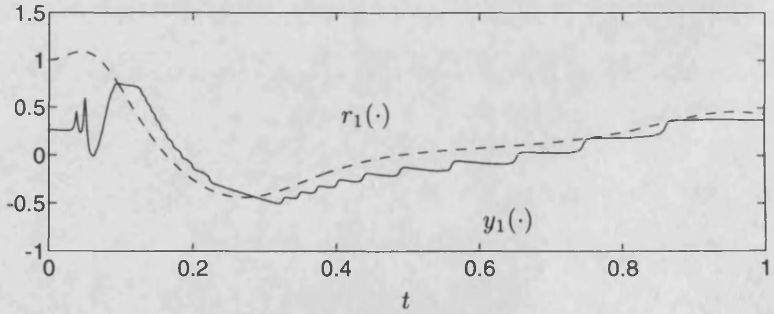


Figure 6-11: The evolution of $y_1(\cdot)$ and $r_1(\cdot)$.

$\alpha$, $\beta$ sense for some $0 < \alpha \leq \beta$. Therefore we change $B^*$, which is now equated to

$$B^* = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$

which is clearly row and column diagonally dominant in the 1, 4 sense.

Other minor changes are made to the system, for purposes of illustration, the function $g(\cdot, \cdot, \cdot)$

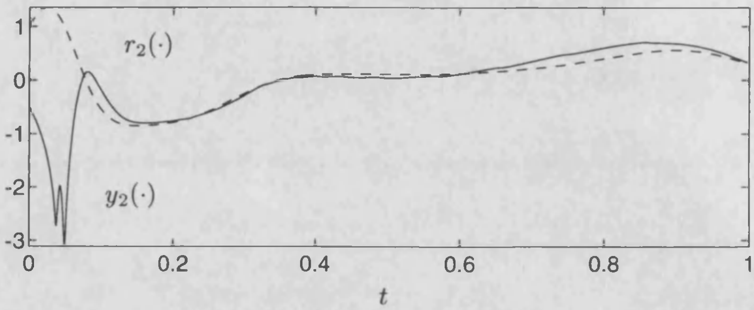


Figure 6-12: The evolution of $y_2(\cdot)$ and $r_2(\cdot)$.

now being given as

$$g(t, z(t), \dot{z}(t)) = \begin{bmatrix} -\alpha_1 z_1^{\frac{1}{3}}(t) + \alpha_2 \cos(\alpha_3 z_1(t)\dot{z}_1(t)) \\ \alpha_4 \sin(\alpha_5 t)|\dot{z}_2(t)|^{\frac{1}{2}} \end{bmatrix},$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (1, 10, 100, 20, 10)$ for this specific example. We specify $\lambda$ to be 0.5, and the function we track is the function $[r_1(t)\ r_2(t)]^T$, where $r_1(\cdot)$ and $r_2(\cdot)$ are generated from the Lorenz equations (6.16).

With initial conditions $z(0) = [0.75\ \ -0.25]$, $\dot{z}(0) = [-0.5\ \ -0.25]$ and $k(0) = 1$ the following results, see Figures $(6-10)$, $(6-11)$ and $(6-12)$ are obtained.

In Figure $(6-10)$, we see the evolution of $k(\cdot)$ is similar to previous examples.

In Figure $(6-11)$ and Figure $(6-12)$ we see $y(\cdot)$ does not exactly track the function $[r_1(\cdot)r_2(\cdot)]^T$, instead tracking it to within the prespecified distance $\lambda = 0.5$.

## 6.5 Perturbed linear systems and output feedback

Similar to Chapter 2 and Chapter 3, the class of system studied so far in this chapter has had the full state available for feedback purposes. We again show how the control strategy of system (6.7), (6.8) and (6.10) can be carried over to the following class of nonlinearly-perturbed linear systems:

$$\left.\begin{aligned} &\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}[u(t) + g(t,\bar{x}(t))] + d(t,\bar{x}(t)),\ \bar{x}(t) \in \mathbb{R}^n,\ u(t) \in \mathbb{R}^m,\ \bar{x}(t_0) = x^0 \\ &\bar{y}(t) = \bar{C}\bar{x}(t), \qquad \bar{y}(t) \in \mathbb{R}^k \end{aligned}\right\} \tag{6.20}$$

under the following assumptions:

**Assumptions**

**(AA1):** For some known $\bar{D} \in \mathbb{R}^{m\times k}$, the triple $(\bar{D}\bar{C}, \bar{A}, \bar{B})$ defines a minimum phase linear system of relative degree one, that is,

$$\operatorname{rank}\begin{bmatrix} sI - \bar{A} & \bar{B} \\ \bar{D}\bar{C} & 0 \end{bmatrix} = n + m, \qquad \forall s \in \bar{\mathbb{C}}_+.$$

where $\bar{\mathbb{C}}_+$ denotes the closed right half complex plane. Also the structural assumptions of **(A1)** hold with $M^{*-1}B^*$ replaced with $\bar{D}\bar{C}\bar{B}$.

**(AA2):** (i) For each $\bar{x} \in \mathbb{R}^n$ the function $g(\cdot,\bar{x})$ is measurable, (ii) for almost all $t \in \mathbb{R}$, $g(t,\cdot)$ is continuous and (iii) there exist a scalar $\mu_1 > 0$ and continuous function $\gamma$ such that, for almost all $t \in \mathbb{R}$,

$$\|g(t,\bar{x})\| \leq \mu_1\gamma(\bar{C}\bar{x}), \qquad \forall \bar{x} \in \mathbb{R}^n$$

**(AA3):** (i) For all $\bar{x} \in \mathbb{R}^n$, $d(\cdot,\bar{x})$ is measurable, (ii) for all $t \in \mathbb{R}$, $d(t,\cdot)$ is continuous and (iii) the function $d$ is bounded in the following manner:

$$\|d(t,\bar{x})\| \leq \mu_2\|\bar{D}\bar{C}\bar{x}\| \qquad \forall (t,\bar{x}) \in \mathbb{R}\times\mathbb{R}^n$$

where $\mu_2$ can be an unknown constant.

Again notice the assumption **(AA1)** requires the existence of a known matrix $\bar{D}$ so that the resulting system, with constructed output $y = \bar{D}\bar{C}\bar{x} \in \mathbb{R}^m$, is minimum phase. The constructed output is needed when the number of outputs exceeds the number of inputs. It is this constructed output which will track the reference signal $r \in W^{1,\infty}(\mathbb{R}, \mathbb{R}^m)$. For this purpose we define the error, $e(\cdot)$, to be as in (6.6).

### 6.5.1 Co-ordinate transformation

For the purposes of analysis we make the same co-ordinate transformation as made in Chapter 2. The resulting system is

$$\left.\begin{array}{l}\dot{w}(t) - L_1 w(t) - L_2 y(t) \in \{v \mid \|v\| \le \mu_2 \|T\| \|y\|\} \\ \dot{y}(t) + L_3 w(t) + L_4 y(t) - \hat{B}u \in \mu^* \mathcal{F}(\bar{y})\end{array}\right\} \tag{6.21}$$

where $\mathcal{F}(\bar{y}) = f(\bar{y})\bar{\mathbb{B}}$, $\mu^* = \mu_1\|\hat{B}\| + \mu_2\|\bar{D}\bar{C}\|$ and $f(\bar{y}) := \gamma(\bar{y}) + \|\bar{y}\|$. Again we find this system, in the absence of the set in the dynamics of $w$, can be identified with (6.5) by setting $M = I$, $B = -\hat{B}$ and $C_{p-1} = -L_4$.

By the minimum-phase condition of **(AA1)** $\sigma(L_1) \subset \mathbb{C}_-$.

### 6.5.2 Adaptive output feedback strategy

Once again the transformation above, transforms the system (6.20) to a format similar to that of (6.5) on which the analysis of previous sections was done. This results in a similar control structure as previously encountered. Let $K(\cdot)$, $\mathcal{O}(\cdot)$, $\mathcal{S}(\cdot)$ and $s(\cdot)$ be as before. Our adaptive feedback strategy is

$$\begin{aligned} &u(t) \in \bar{\Psi}_1(t, \bar{y}(t)) := k^2(t)[f(\bar{y}(t)) + \|\bar{D}\bar{y}(t) - r(t)\| + 1]\mathcal{O}_+(k(t))\mathcal{S}(s(k(t)))\psi(\bar{D}\bar{y}(t) - r(t)), \\ &\dot{k}(t) = \|\bar{D}\bar{y}(t) - r(t)\|[f(\bar{y}(t)) + \|\bar{D}\bar{y}(t) - r(t)\| + 1], \qquad k(0) = k^0 \end{aligned}$$

in the case of asymptotic tracking when $m = 2$, or

$$\begin{aligned} &u(t) \in \bar{\Psi}_2(t, \bar{y}(t)) := -k(t)[f(\bar{y}(t)) + \|\bar{D}\bar{y}(t) - r(t)\| + 1]K(k(t))\psi(\bar{D}\bar{y}(t) - r(t)), \\ &\dot{k}(t) = \|\bar{D}\bar{y}(t) - r(t)\|[f(\bar{y}(t)) + \|\bar{D}\bar{y}(t) - r(t)\| + 1], \qquad k(0) = k^0 \end{aligned}$$

in the case of asymptotic tracking for general $m$, or

$$\begin{aligned} &u(t) = \bar{\Psi}_3(t, \bar{y}(t)) := \\ &\qquad -k(t)[f(\bar{y}(t)) + \|\bar{D}\bar{y}(t) - r(t)\| + d_\lambda(\bar{D}\bar{y}(t) - r(t)) + 1]K(k(t))\psi_\lambda(\bar{D}\bar{y}(t) - r(t)), \\ &\dot{k}(t) = d_\lambda(\bar{D}\bar{y}(t) - r(t))[f(\bar{y}(t)) + \|\bar{D}\bar{y}(t) - r(t)\| + d_\lambda(\bar{D}\bar{y}(t) - r(t)) + 1], \qquad k(0) = k^0 \end{aligned}$$

in the case of $\lambda$-tracking, for general $m$.

Analogous theory to that in previous sections of this chapter proves this strategy is a universal control for tracking objectives for (6.20), with the observations

- the constructed output $\bar{D}\bar{y}(t)$ tracks the reference signal,
- where the boundedness of $w(\cdot)$ is considered, we note for $t \in [t_0, \omega)$

$$\|w(t)\| \le \|\exp(L_1(t - t_0))\|\|w(t_0)\| + \int_{t_0}^{t} \|\exp(L_1(t - s))\|[\|L_2\|(c_1 + \|r\|_{1,\infty}) + c_2]\, ds$$

for some constants $c_1$ and $c_2$, where $\sigma(L_1) \subset \mathbb{C}_-$

# Chapter 7

# Adaptive tracking of nonlinear systems

## 7.1 Introduction

As with the previous chapter, the control objectives of asymptotic and $\lambda$-tracking shall be considered. Again the reference signals to be tracked will be from the class $W^{1,\infty}(\mathbb{R}, \mathbb{R}^m)$. For a summary of tracking and the advantages of $\lambda$-tracking see the introduction of the previous chapter.

The classes of system to be considered shall be similar to those of Chapters 4 and 5, with only minor alterations to the structural assumptions. The matrix theory of the previous chapters will be used to help overcome the uncertainties of the input-connection matrix-valued function $H(\cdot,\cdot,\cdot)$.

## 7.2 A class of multi-input, multi-output nonlinear systems

The general class $C$, of nonlinear system to be studied for the problem of the tracking objective is of the same general form as that of Chapters 4 and 5; namely

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), y(t)) \\ \dot{y}(t) &= g(t, x(t), y(t)) + H(t, x(t), y(t))u \end{aligned} \tag{7.1}$$

where $x(t) \in \mathbb{R}^n$, $y(t)$, $u(t) \in \mathbb{R}^m$, $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $g : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ and $H : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{m\times m}$. We shall explicitly construct controls for both the case where $m = 1$, and the case where $m$ is general. The general structural assumptions made on the system are similar to those of Chapters 4 and 5. They are:

**General Structural Assumptions.**

**(G1):** The functions $f(\cdot,\cdot,\cdot)$, $g(\cdot,\cdot,\cdot)$ and $H(\cdot,\cdot,\cdot)$ are continuous.

**(G2):** There exists an $\epsilon > 0$, a known continuous function $\gamma : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}_+$ with $\gamma(t,y) \geq \epsilon$ for all $(t,y) \in \mathbb{R}\times\mathbb{R}^m$, and scalars $\alpha, \beta \in \mathbb{R}_+$ (not necessarily known) with $0 < \alpha \leq \beta < \infty$ such that

(ia) in the case $m = 1$, $H(\cdot,\cdot,\cdot)$ is a continuous real-valued function satisfying

$$\alpha\gamma(t,y) \leq |H(t,x,y)| \leq \beta\gamma(t,y), \qquad \forall (t,x,y) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m,$$

(ib) in the case of general $m$, the matrix-valued function $H(\cdot,\cdot,\cdot)$ is essentially row and column diagonally dominant in the $\alpha\hat{\gamma}$, $\beta\hat{\gamma}$ sense, where $\hat{\gamma}(t,x,y) = \gamma(t,y)$ for all $(t,x,y) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$,

(ii) $K \subset \mathbb{R}^m$ bounded $\implies \gamma(\mathbb{R}, K)$ bounded.

**(G3):** The point $(x,y) = (0,0)$ is the unique stationary point of the system (7.1) in the absence of control.

**(G4):** The function $f(\cdot,\cdot,0)$ is time invariant, ie. $f(t,x,0) = \bar{f}(x)$ for all $(t,x) \in \mathbb{R} \times \mathbb{R}^n$ for some continuous $\bar{f}(\cdot)$.

For an explanation of the general form of these general structural assumptions see Chapters 4 and 5.

In this tracking case, we shall look only at two different but non-exclusive subclasses, $\mathcal{C}_1$ and $\mathcal{C}_2$ of the class $\mathcal{C}$. These will essentially be the same as the second and third subclass considered in Chapters 4 and 5. This is primarily due to the way in which the proof of the first subclass progresses; the problem arises from a Liapunov type function being constructed for both the $x$ and $y$ systems simultaneously. In this tracking case this would result in an unwanted constant appearing, similar to the way in which $c^o$ appears in (7.13). This problem is overcome in Theorem 7.5.3, using Lemma 7.5.2, by considering the two systems $x$ and $y$ separately. This method cannot be used in the Isidori and Byrnes inspired subclass, since we do not have an equivalent lemma to Lemma 7.5.2 to cope with the $\|x\|$ terms when considering the $y$ system.

## 7.3 The controls for tracking objectives

Here we give the universal control strategies that will achieve the objectives of asymptotic and $\lambda$-tracking for both the cases where $m = 1$ and general $m$. The function to be tracked is permitted to be any function from the class of reference signals $W^{1,\infty}(\mathbb{R}, \mathbb{R}^m)$. Recall the definition of the error, (6.6), as this will be used throughout the chapter.

### 7.3.1 A universal control for the asymptotic tracking objective

Here we give the universal controls that achieve the objective of asymptotically tracking a given reference signal. We give controls, distinguishing between the cases $m = 1$ and general $m$. In the general $m$ case we use the cycling mechanism (3.4) which cycles through the unmixing set $I_m^E$. Refer to Chapter 5 for an alternative, smaller set, which could be used in this cycling function if certain conditions are met. A universal control that achieves the control objective of asymptotic tracking is constructed in the following way. Let $w = (x, e, k)$, for $i = 1$, 2, define $\psi_{1,i}^*(\cdot, \cdot)$ as

$$\psi_{1,i}^*(t, w) := \begin{cases} 1 + \rho(y(t)), & i = 1 \\ 1 + \|e\| + \rho(y(t)) + \|e\|\rho^2(e)[1 + \|e\|^2\rho^2(e)], & i = 2 \end{cases}, \tag{7.2}$$

where $\rho(\cdot)$ is a known continuous function with properties dependent on the subclass of $\mathcal{C}$ under consideration. In the bounded input, bounded output case $i = 1$, and in the exponentially stable zero dynamics case, $i = 2$. With this we are now able to define the controls. For $i = 1$, 2, and $j = 1$, 2, define $\Psi_{1,i,j}(\cdot, \cdot)$ as

$$\Psi_{1,i,j}(t, w) := \begin{cases} \nu(k)\psi_{1,i}^*(t, w)\psi(e), & j = 1 \\ -k\psi_{1,i}^*(t, w)K(k)\psi(e), & j = 2. \end{cases} \tag{7.3}$$

$\psi(\cdot)$ is the upper-semicontinuous set-valued map (2.7) which has compact and convex values, $K(\cdot)$ is the function (3.4) and $\nu(\cdot)$ is a scaling invariant Nussbaum function, see (1.2). When $m = 1$, $j = 1$, and when $m$ is general, $j = 2$.

**Note:** The subscript 1 in the notation $\psi_{1,i}^*$ and $\Psi_{1,i,j}$, indicates the use in the control of the asymptotic objective, as opposed to the notation in the next subsection where a subscript 2 will replace this 1 to indicate the use in the control of the $\lambda$-tracking objective. The $i$ subscript indicates the subclass under consideration and the $j$ subscript is an indication of whether the dimension of the input and output is 1 or of dimension greater than 1.

### 7.3.2 A universal control for the $\lambda$-tracking objective

Here we give the universal controls that achieve the objective of $\lambda$-tracking a given reference signal. This is where the error asymptotically tends to the ball of radius $\lambda$. See the introduction of the previous chapter for the benefits of $\lambda$-tracking. Again we give controls, distinguishing between the cases $m = 1$ and general $m$; the latter case using the cycling mechanism (3.4).

With $d_\lambda(\cdot)$ as defined in (6.9), define $V^* : \mathbb{R}^m \to \bar{\mathbb{R}}_+$ as the $C^1$ function

$$V^*(e) := \frac{1}{2}d_\lambda^2(e), \tag{7.4}$$

with derivative

$$\nabla V^*(e) := \begin{cases} d_\lambda(e)e/\|e\|, & \|e\| > \lambda \\ 0, & \|e\| \le \lambda. \end{cases} \tag{7.5}$$

Recall the definition of $\psi_\lambda(\cdot)$ of the previous chapter. A universal control that achieves the control objective of $\lambda$-tracking is given as the following: let $w = (x, e, k)$, for $i = 1, 2$, define $\psi^*_{2,i}(\cdot,\cdot)$ as

$$\psi^*_{2,i}(t,w) := \begin{cases} 1 + d_\lambda(e) + \rho(y), & i = 1 \\ 1 + \rho(y) + d_\lambda(e) + d_\lambda(e)\rho^2(\nabla V^*(e))[1 + d_\lambda^2(e)\rho^2(\nabla V^*(e))], & i = 2, \end{cases} \tag{7.6}$$

where $\rho(\cdot)$ is a known continuous function with properties dependent on on the subclass of $\mathcal{C}$. With this definition we are able to define the control. For $i = 1, 2$, and $j = 1, 2$, define $\Psi_{2,i,j}(\cdot,\cdot)$ as

$$\Psi_{2,i,j}(t,w) := \begin{cases} \nu(k)\psi^*_{2,i}(t,w)\psi_\lambda(e), & j = 1 \\ -k\psi^*_{2,i}(t,w)K(k)\psi_\lambda(e), & j = 2. \end{cases} \tag{7.7}$$

$K(\cdot)$ is the function (3.4) and $\nu(\cdot)$ is a scaling invariant Nussbaum function, see (1.2). Again, $i = 1$ for the bounded input, bounded output case, $i = 2$ for the exponentially stable zero dynamics case, $j = 1$ for $m = 1$ and $j = 2$ for general $m$. Note the control is continuous, which is an advantage of this control objective.

### 7.3.3 Embedding the system into a differential inclusion

Therefore the class of system $\mathcal{C}$ can be embedded in the following differential inclusion:

$$\dot{w}(t) \in F(t, w(t)), \qquad w(t_0) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_+ \tag{7.8}$$

with $F(t,w) := F_1(t,w) \times F_2(t,w) \times F_3(t,w)$, where

$$F_1(t,w) := \{f(t,x,y)\}$$

$$F_2(t,w) := \begin{cases} \{g(t,x,y) - \xi + H(t,x,y)u \mid \|\xi\| \le \|r\|_{1,\infty},\ u \in \Psi_{1,i,j}(t,w)\}, \\ \qquad \text{for asymptotic tracking} \\ \{g(t,x,y) - \xi + H(t,x,y)\Psi_{2,i,j}(t,w) \mid \|\xi\| \le \|r\|_{1,\infty}\}, \\ \qquad \text{for } \lambda\text{-tracking} \end{cases}$$

$$F_3(t,w) := \begin{cases} \{\|e\|\psi^*_{1,i}(t,w)\gamma(t,y(t))\}, & \text{for asymptotic tracking} \\ \{d_\lambda(e)\psi^*_{2,i}(t,w)\gamma(t,y(t))\}, & \text{for } \lambda\text{-tracking} \end{cases}$$

$\gamma(\cdot,\cdot)$ is the known continuous function of general assumption **(G2)**. Here

$$i = \begin{cases} 1, & \text{for the bounded-input, bounded-output subclass} \\ 2, & \text{for the exponentially stable zero dynamics subclass,} \end{cases}$$

and

$$j = \begin{cases} 1, & \text{for } m = 1 \\ 2, & \text{for general } m. \end{cases}$$

A solution to this initial value problem exists on a maximal interval of existence by Theorem $A.3.1$ and Theorem $A.3.2$.

## 7.4 A system with a bounded input/bounded output subsystem

The first subclass $\mathcal{C}_1$ of the class $\mathcal{C}$ to be studied has the key feature of having bounded-input, bounded-output in the sub-state $x$ dynamics, as in Chapters 4 and 5. The structural assumptions are similar to the equivalent subclass of these chapters.

**Structural Assumptions:**

**(A2.1):** The general assumptions **(G1)** to **(G4)** hold.

**(A2.2):** Bounded input (i.e. $\|y\|_\infty < \infty$) for the system $\dot{x}(t) = f(t, x(t), y(t))$ implies bounded output (i.e. $\|x\|_\infty < \infty$).

**(A2.3):** For some known continuous function $\rho : \mathbb{R}^m \to \bar{\mathbb{R}}_+$, the function $g(\cdot,\cdot,\cdot)$ is continuous with

$$\|g(t,x,y)\| \le \gamma_1 \rho(y), \qquad \forall (t,x,y) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$$

where $\gamma_1 > 0$ is a possibly unknown constant.

**(A2.4):** The function $x \mapsto \bar{f}(x) (= f(t,x,0))$ has the following properties:

- $x = 0 \iff \bar{f}(x) = 0$.
- The origin is globally-asymptotically stable for the system $\dot{x}(t) = \bar{f}(x(t))$.
- $x \mapsto \bar{f}(x)$ is locally Lipschitz.

**(A2.5):** There exists a continuous function $f^\dagger : \mathbb{R}^n \times \mathbb{R}^m \to \bar{\mathbb{R}}_+$ with $f^\dagger(x,0) = 0$ for all $x \in \mathbb{R}^n$ such that

$$\|f^*(t,x,y)\| \le f^\dagger(x,y), \qquad \forall (t,x,y) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m.$$

For the reasoning behind these structural assumptions, see Chapters 4 and 5. Note that we longer require $\rho(\cdot)$ to satisfy $\rho(y) \geq \|y\|^p$ for some $p$ with $0 \leq p < \infty$.

### 7.4.1 The asymptotic tracking control objective

With theory established in the previous chapters we can immediately give a proof establishing (i) the maximal interval of existence of solutions of (7.8) is the infinite interval $[t_0, \infty)$, (ii) the adaptive gain $k(\cdot)$ tends to a finite limit, (iii) $x(\cdot)$ is bounded and (iv) the size of the error tends to zero as $t$ tends to infinity.

**Theorem 7.4.1** *Let $w : [t_0, \omega) \to \mathbb{R}^{n+m+1}$ be a solution to the differential inclusion (7.8) on its maximal interval of existence, then*

*(i) $\omega = \infty$.*

*(ii) $\lim_{t\to\infty} k(t)$ exists and is finite.*

*(iii) $w(\cdot)$ is bounded.*

*(iv) $e(t) \to 0$ as $t \to \infty$.*

**Proof:** Firstly we shall show $k(\cdot)$ is bounded, which will also give the boundedness of $e(\cdot)$. Consider the function

$$E : e \mapsto \frac{1}{2}\|e\|^2$$

For all $(t, w) \in \mathbb{R} \times \mathbb{R}^{n+m+1}$, $m > 1$,

$$\begin{aligned}\langle e, \eta\rangle &\leq \langle e, g(t,x,y)\rangle + \max_{u\in\Psi_{1,1,2}(t,w)}\{\langle e, H(t,x,y)u\rangle\} + \|r\|_{1,\infty}\|e\| \\ &\leq \epsilon^{-1}(\gamma_1 + \|r\|_{1,\infty})\|e\|[1+\rho(y)]\gamma(t,y) - k\nu(k)[1+\rho(y)]\gamma(t,y)\|e\|\end{aligned}$$

for all $\eta \in F_2(t,w)$, using Lemma 5.5.1 and similar arguments as in Theorem 5.6.1, with $\nu(k)$ defined as in Lemma 3.5.2. In the case $m = 1$, the techniques used in Theorem 4.5.2 give, for all $(t, w) \in \mathbb{R} \times \mathbb{R}^{n+m+1}$,

$$\begin{aligned}e\eta &= eg(t,x,y) + \max_{u\in\Psi_{1,1,1}(t,w)}\{eH(t,x,y)u\} + \|r\|_{1,\infty}|e| \\ &\leq \epsilon^{-1}(\gamma_1 + \|r\|_{1,\infty})|e|[1+\rho(y)]\gamma(t,y) + \mathcal{N}(k)[1+\rho(y)]\gamma(t,y)|e|\end{aligned}$$

for all $\eta \in F_2(t,w)$, where $\mathcal{N}(\cdot)$ is defined as in (4.11). Recalling

$$\dot{k}(t) = \|e(t)\|[1+\rho(y(t))]\gamma(t,y(t))$$

and defining $\gamma^* := \epsilon^{-1}(\gamma_1 + \|r\|_{1,\infty})$ we get for $t \in [t_0, \omega)$,

$$\begin{array}{ll} 0 \leq E(e(t)) \leq E(e(t_0)) + \gamma^*(k(t) - k(t_0)) + \int_{k(t_0)}^{k(t)} \mathcal{N}(k)\, dk, & m = 1 \\ 0 \leq E(e(t)) \leq E(e(t_0)) + \gamma^*(k(t) - k(t_0)) - \int_{k(t_0)}^{k(t)} \nu(k) k\, dk, & m > 1 \end{array} \tag{7.9}$$

By an analogous arguments as found in Theorem 4.5.2 and Theorem 5.6.1, by assuming $k(\cdot)$ is unbounded, a contradiction arises. Therefore $k(\cdot)$ is bounded.

From $k(\cdot)$ being bounded and (7.9) the boundedness of $e(\cdot)$ and hence $y(\cdot)$ is established. From assumption **(A2.2)** we get the boundedness of $x(\cdot)$. This in turn gives $\omega = \infty$ by Theorem $A$.3.3 in the appendix.

Now we show $e(t) \to 0$ as $t \to \infty$. We do this by showing $e(\cdot) \in L^1$, $\dot{e}(\cdot) \in L^\infty$ and then using Lemma $D$.2.1 in the appendix. Since $\dot{k}(t) = \|e(t)\|[1 + \rho(y(t))]\gamma(t, y(t)) \geq \epsilon\|e(t)\|$, and $k(\cdot)$ is bounded, $e(\cdot) \in L^1$. The boundedness of $\dot{e}(\cdot)$ is obtained directly from the dynamics of $e$ in (7.8), since $g(\cdot,\cdot,\cdot)$ is bounded by the continuous function $\rho(\cdot)$ and $\|H(\cdot,\cdot,\cdot)\|$ is bounded by $\gamma(\cdot,\cdot)$, where $\gamma(\mathbb{R}, K)$ is bounded for bounded $K \subset \mathbb{R}^m$, the control is upper-semicontinuous and the arguments of the functions are bounded. So indeed $e(t) \to 0$ as $t \to \infty$ in the case $m > 1$. A analogous argument gives the same result in the case $m = 1$. □

**Example**

Now we consider an example, $n = 2$, $m = 3$, that satisfies the structural assumptions of this

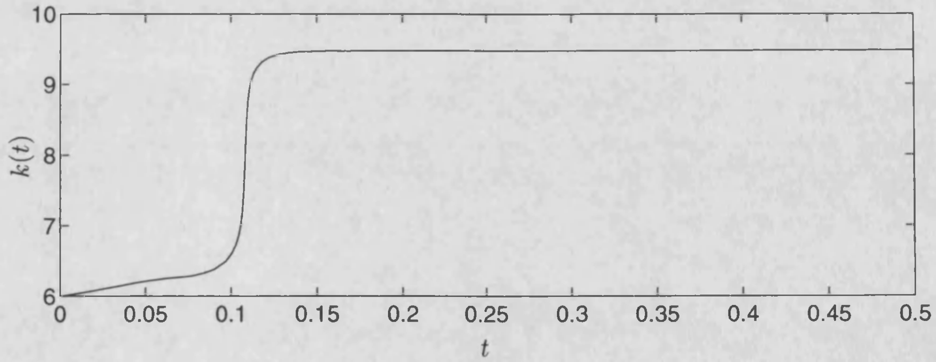


Figure 7-1: The evolution of $k(\cdot)$.

section with $\rho(y) = \exp(\|y\|)$ and $\gamma(t, y) = 1$. The specific member of this class we shall consider is

$$\dot{x}(t) = -\|x(t)\|x(t) + \exp(y_3^3(t) - 1) \begin{bmatrix} \sin(t) & 0 \\ 0 & \cos(t) \end{bmatrix} x(t)$$

$$\dot{y}(t) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + y(t) + u$$

The function we shall track is the function $[r_1(\cdot), r_2(\cdot), r_3(\cdot)]^T$, as generated by (6.16). With

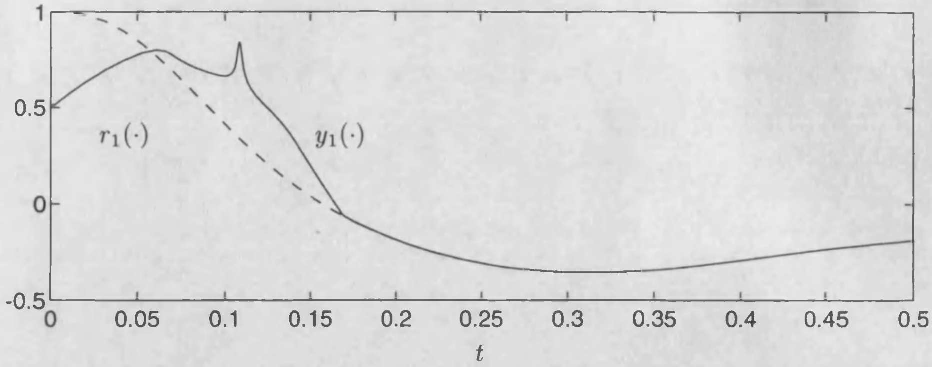


Figure 7-2: The evolution of $y_1(\cdot)$ and $r_1(\cdot)$.

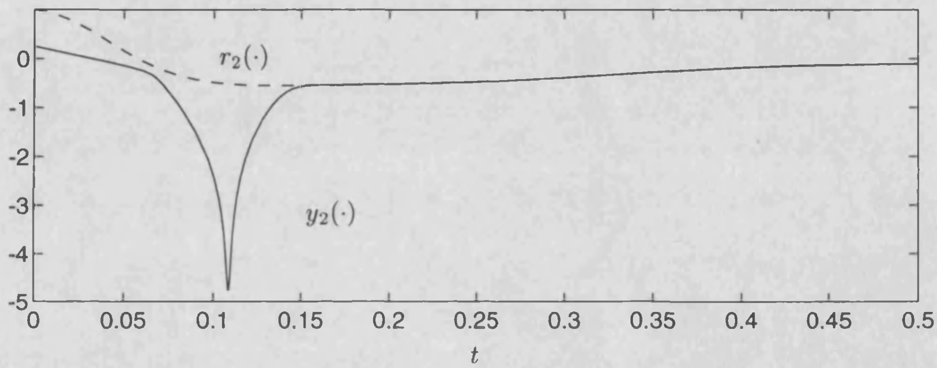


Figure 7-3: The evolution of $y_2(\cdot)$ and $r_2(\cdot)$.

initial conditions $x(0) = [2\ -1]$, $y(0) = [0.5\ 0.25\ 0]$ and $k(0) = 2$, the results in Figures $(7-1)$, $(7-2)$, $(7-3)$ and $(7-4)$, are obtained.

Again, in Figure $(7-1)$, we find the evolution of $k(\cdot)$ is as expected.

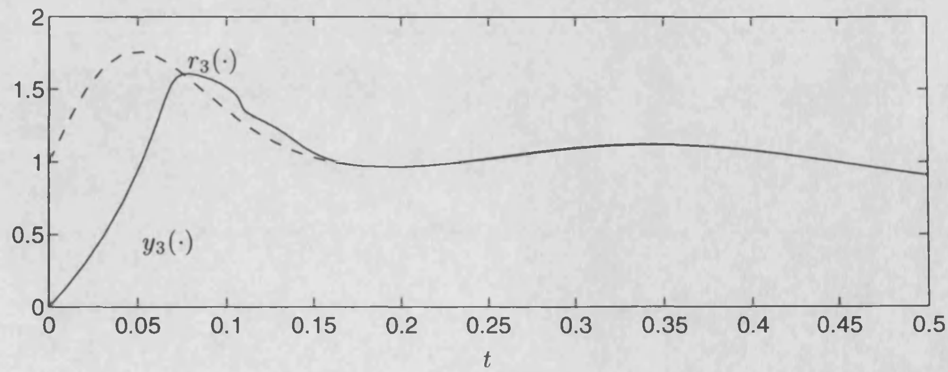


Figure 7-4: The evolution of $y_3(\cdot)$ and $r_3(\cdot)$.

In Figure $(7-2)$, $y_1(\cdot)$ tracks $r_1(\cdot)$ as $k(\cdot)$ converges.

In Figure $(7-3)$, $y_2(\cdot)$ tracks $r_2(\cdot)$ as $k(\cdot)$ converges.

In Figure $(7-4)$, $y_3(\cdot)$ tracks $r_3(\cdot)$ as $k(\cdot)$ converges.

### 7.4.2 The $\lambda$-tracking control objective

Now we consider the control objective of $\lambda$-tracking for this bounded input, bounded output subsystem case. The same structural assumptions are imposed as in the asymptotic tracking case. Consequently the proof that establishes, (i) the maximal interval of existence of a solution of the differential inclusion (7.8) is the infinite interval $[t_0, \infty)$, (ii) the adaptive gain $k(\cdot)$ tends to a finite limit (iii) the state $x(\cdot)$ is bounded and (iv) the size of the error asymptotically tends to the interval $[0, \lambda]$, follows the same lines as Theorem 7.4.1. As would be expected the control is different, namely $\Psi_{2,1,j}(\cdot,\cdot)$, where $j = 1$ for $m = 1$ and $j = 2$ for general $m$.

**Theorem 7.4.2** *Let $w : [t_0, \omega) \to \mathbb{R}^{n+m+1}$ be a solution to the differential inclusion (7.8) on its maximal interval of existence, then*

*(i)* $\omega = \infty$,

*(ii)* $\lim_{t\to\infty} k(t)$ *exists and is finite,*

*(iii)* $w(\cdot)$ *is bounded,*

*(iv)* $d_\lambda(e) \to 0$ *as* $t \to \infty$.

**Proof:** Recall the definition of $V^*(\cdot)$, (7.4), and its derivative (7.5). For all $(t, w) \in \mathbb{R}\times\mathbb{R}^{n+m+1}$, $e \neq 0$,

$$\begin{aligned}\langle \nabla V^*(e), \eta\rangle &\leq d_\lambda(e)\|e\|^{-1}[\langle e, g(t,x,y)\rangle + \|r\|_{1,\infty}\|e\| + \langle e, H(t,x,y)\Psi_{2,1,j}(t,w)\rangle] \\ &\leq c d_\lambda(e)(1+\rho(y)) - k\nu(k) d_\lambda(e)[1 + d_\lambda(e) + \rho(y)]\gamma(t,y),\end{aligned}$$

for all $\eta \in F_2(t, w)$, where $j$ depends on the dimension $m$, $c := \gamma_1 + \|r\|_{1,\infty}$, $\nu(\cdot)$ is the function defined in Lemma 3.5.2 and using Lemma 5.5.1, noting $d_\lambda(\cdot)$ compensates for $\psi_\lambda(\cdot)$ when $\|e\| \leq \lambda$. Therefore

$$0 \leq V^*(e(t)) \leq V^*(e(t_0)) + \epsilon^{-1}c(k(t) - k(t_0)) - \int_{k(t_0)}^{k(t)} k\nu(k)\,dk, \tag{7.10}$$

for general $m$, and in the case $m = 1$, the last term is replaced with $+\int_{k(t_0)}^{k(t)} \mathcal{N}(k)\,dk$, where $\mathcal{N}(\cdot)$ is the continuous Nussbaum function as defined in (4.11).

By the usual argument, see Theorem 4.5.2 and Theorem 5.6.1, $k(\cdot)$ is bounded, and therefore by (7.10) $d_\lambda(\cdot)$ is bounded, which in turn gives the boundedness of $e(\cdot)$ and $y(\cdot)$. Structural assumption **(A2.2)** gives the boundedness of $x(\cdot)$. Therefore by Theorem $A.3.3$ $\omega = \infty$.

We now show that $d_\lambda(e(t)) \to 0$ as $t \to \infty$. We do this by showing $V^*(e(t)) \to 0$ as $t \to \infty$. This can be done by the usual argument, using Lemma $D.2.1$. $V^*(e(\cdot)) \in L^1$ since $k(\cdot)$ is bounded and $\dot{k}(t) \geq \epsilon^{-1} d_\lambda^2(e(t))$. $\frac{dV^*(e(\cdot))}{dt} \in L^\infty$, since $\nabla V^*(\cdot)$ is continuous and $e(\cdot)$ is bounded, and $\dot{e}(\cdot)$ is bounded directly from (7.8) using the boundedness of $y(\cdot)$ and $x(\cdot)$ and the fact that $\rho(\cdot)$ is continuous and $\gamma(\cdot,\cdot)$ satisfies $\gamma(\mathbb{R}, K)$ is bounded for bounded $K \subset \mathbb{R}^m$. Therefore $V^*(e(t)) \to 0$ as $t \to \infty$, which implies $d_\lambda(e(t)) \to 0$ as $t \to \infty$. □

**Example**

Here we consider an example that satisfies the structural assumptions of this section with $\rho(y) =$

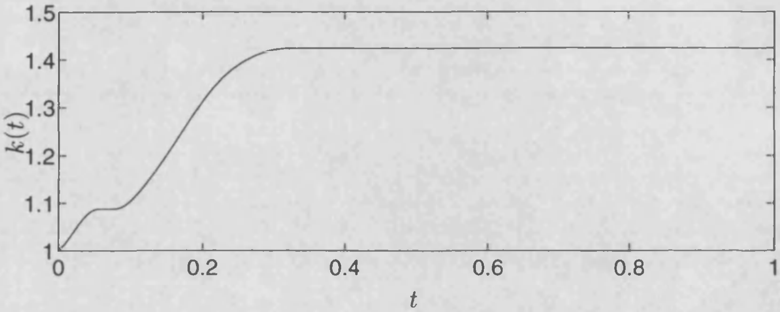


Figure 7-5: The evolution of $k(\cdot)$.

$\exp(\|y\|)$ and $\gamma(t, y) = 1$. The specific example we consider is with $n = 2$ and $m = 2$:

$$\dot{x}(t) = -\|x(t)\|x(t) + \exp(y_2^3(t) - 1) \begin{bmatrix} \sin(t) & 0 \\ 0 & \cos(t) \end{bmatrix} x(t)$$

$$\dot{y}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + y(t) + u$$

The function we shall track is the function $[r_1(t)\ r_2(t)]^T$, generated from (6.16), to within the distance $\lambda = 0.5$. With initial conditions $x(0) = [2\ -1]$, $y(0) = [0.5\ 0.25]$ and $k(0) = 1$, the results in Figures $(7-5)$, $(7-6)$ and $(7-7)$, are obtained.

As seen in Figure $(7-5)$, the evolution of $k(\cdot)$ follows the standard pattern. We see the evolution 'levels out' twice. The first time, although the error is within the prespecified distance $\lambda$, the

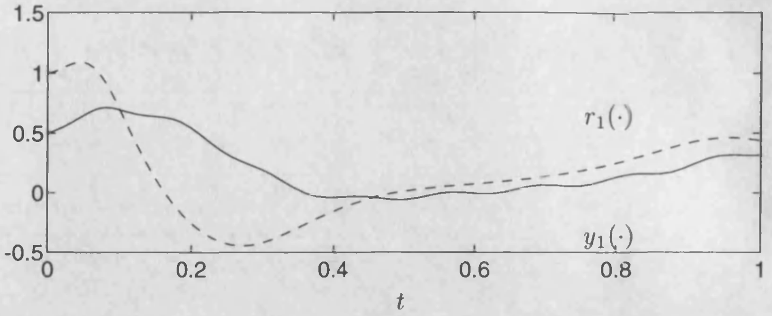


Figure 7-6: The evolution of $y_1(\cdot)$ and $r_1(\cdot)$.

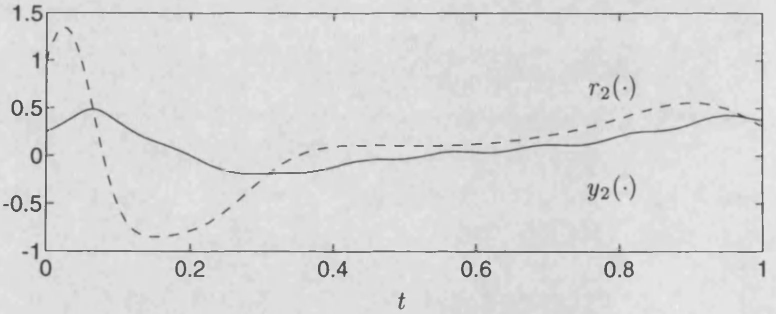


Figure 7-7: The evolution of $y_2(\cdot)$ and $r_2(\cdot)$.

value of $k$ is not large enough.

In Figure $(7-6)$ the evolution of $y_1(\cdot)$ tracks $r_1(t)$ to within the prespecified distance $\lambda = 0.5$, as $k(\cdot)$ converges.

In Figure $(7-7)$ the evolution of $y_2(\cdot)$ tracks $r_2(t)$ to within the prespecified distance $\lambda = 0.5$, as $k(\cdot)$ converges.

The output does not appear to be tracking the reference signal very well, but this is due to $\lambda = 0.5$. Note, $\lambda > 0$ can be made arbitrarily small.

## 7.5 Exponentially-asymptotically stable zero-dynamics

Again the final subclass $\mathcal{C}_2$ of the class $\mathcal{C}$ to be studied, has the key feature of having exponentially stable zero dynamics. It is classified by structural assumptions, **(A3.1)** to **(A3.4)** below.

**Structural Assumptions:**

**(A3.1):** The general assumptions **(G.1)** to **(G.4)** hold.

**(A3.2):** The function $x \mapsto \bar{f}(x)$ satisfies the following properties:

- $\bar{f}(x) = 0 \iff x = 0$,
- $x \mapsto \bar{f}(x)$ is globally Lipschitz.
- There exists a $c > 0$ and a $K > 0$ such that for $x^0 \in \mathbb{R}^n$, the solution $x(t)$ to $\dot{x}(t) = \bar{f}(x(t))$ with $x(t_0) = x^0$ satisfies

$$\|x(t)\| \leq K \exp(-c(t-t_0))\|x^0\|, \qquad \forall t \geq t_0,$$

ie. the origin is exponentially stable.

For some known continuous function $\rho : \mathbb{R}^m \to \bar{\mathbb{R}}_+$, the following hold.

**(A3.3):** The residue function $f^*(\cdot,\cdot,\cdot)$, has the following bound:

$$\|f^*(t,x,y)\| \leq \gamma_1(1+\|x\|^{\delta_1})h^*(y), \qquad \forall (t,x,y) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$$

where $\gamma_1 > 0$ is a possibly unknown constant and $0 \leq \delta_1 < 1$. $h^* : \mathbb{R}^m \to \bar{\mathbb{R}}_+$ is continuous and satisfies the following:

- for all $r$ such that $1 \leq r < \infty$

$$(h^*(y))^r \leq \gamma_{2,r}\|y\|\rho(y), \qquad \forall y \in \mathbb{R}^m,$$

where $\gamma_{2,r}$ is a possibly unknown constant that can depend on $r$,

- and

$$h^*(y_1 + y_2) \leq \gamma_6(h^*(y_1) + h^*(y_2)), \qquad \forall y_1,\, y_2 \in \mathbb{R}^m.$$

**(A3.4):** The function $g(\cdot,\cdot,\cdot)$ is in the class of functions such that,

$$\|g(t,x,y)\| \leq \gamma_3(1 + \|x\|^{\delta_2})l(y), \qquad \forall (t,x,y) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m,$$

where $\gamma_3 > 0$ is a possibly unknown constant and $0 \leq \delta_2 < 2$. The function $l : \mathbb{R}^m \to \bar{\mathbb{R}}_+$ satisfies, for all $r$ such that $1 \leq r < \infty$,

$$(l(y))^r \leq \gamma_{4,r}\rho(y), \qquad \forall y \in \mathbb{R}^m$$

where $\gamma_{4,r}$ is a possibly unknown constant that can depend on $r$.

Note the difference in structural assumption **(A3.3)** when compared with the analogous assumption from Chapters 4 and 5. In a linear based system this extra condition has been used, being satisfied automatically. The reader can think of $h^*(\cdot)$ as the function $(\|y\| + \|y\|^\lambda)$, for some $\lambda \geq 1$ (with $\rho(y) = \exp(\|y\|)$). Such an $h^*(\cdot)$ satisfies the requirements, by the following lemma.

**Lemma 7.5.1** *If $\alpha,\ \beta \in \mathbb{R}_+$, then for $r \geq 0$*

$$(\alpha + \beta)^r \leq c^*(\alpha^r + \beta^r)$$

*where $c^*$ is some positive constant.*

**Proof:** The function $(\alpha,\beta) \mapsto (\alpha^r + \beta^r)$ is a homogeneous function of order $r$, see Appendix B for details. Therefore by Lemma $B.2.1$ there exists a constant $K_1 > 0$ such that

$$\begin{aligned}(\alpha^r + \beta^r) &\geq K_1\|(\alpha,\beta)\|^r = K_1(\alpha^2 + \beta^2)^{r/2} \\ &\geq c^\dagger(\alpha + \beta)^r, \end{aligned} \tag{7.11}$$

where $c^\dagger > 0$ is some positive constant. (The last inequality comes from the fact $(\alpha + \beta)^2 \leq 2(\alpha^2 + \beta^2)$). This proves the result. □

Again we make use of the exponential decay of the Liapunov function $V(\cdot)$, [83], along the solution of the zero dynamics. We give a further lemma concerning an integral involving $V(\cdot)$, which is useful in this setting.

**Lemma 7.5.2** *Let $V : \mathbb{R}^n \to \bar{\mathbb{R}}_+$ be a locally Lipschitz function, $x : [t_0, \omega) \to \mathbb{R}^n$ an absolutely continuous function, $\theta : \mathbb{R} \to \bar{\mathbb{R}}_+$ and $\beta : \mathbb{R} \to \bar{\mathbb{R}}_+$ locally integrable functions. If*

$$V^+(x(t);\dot{x}(t)) + cV(x(t)) \leq \theta(t)\beta(t) + d, \qquad \textit{for almost all } t \in [t_0,\omega), \tag{7.12}$$

*where $c$ and $d$ are positive constants, then for $t \in [t_0, \omega)$*

$$\int_{t_0}^{t} \theta(s)V^2(x(s))\,ds \le a_1 \int_{t_0}^{t} \theta(s)\,ds + a_2\sqrt{\int_{t_0}^{t} \theta^2(s)\,ds}\left[\sqrt{\int_{t_0}^{t} (\theta(s)\beta(s))^2\,ds} + \sqrt{\int_{t_0}^{t} (\theta(s)\beta(s))^4\,ds}\right],$$

*where $a_1$ and $a_2$ are constants.*

**Proof:** From (7.12) we obtain

$$2\exp(cs)V(x(s))V^+(x(s);\dot{x}(s)) + 2c\exp(cs)V^2(x(s)) \le 2\exp(cs)(\theta(s)\beta(s) + d)V(x(s)).$$

By recalling Young's inequality (4.9) we get

$$2\exp(cs)V(x(s))V^+(x(s);\dot{x}(s)) + c\exp(cs)V^2(x(s)) \le \frac{1}{c}\exp(cs)(\theta(s)\beta(s) + d)^2.$$

Therefore

$$\int_{t_0}^{t} \theta(s)V^2(x(s)) \le \int_{t_0}^{t} \exp(-c(s-t_0))\theta(s)V^2(x(t_0)) + \frac{1}{c}\int_{t_0}^{t} \exp(-cs)\theta(s)\int_{t_0}^{s} \exp(c\tau)(\theta(\tau)\beta(\tau) + d)^2\,d\tau ds.$$

Now for locally integrable $\psi(\cdot)$

$$\int_{t_0}^{t} \exp(-cs)\theta(s)\int_{t_0}^{s} \exp(c\tau)\psi(\tau)\,d\tau ds \le \left(\int_{t_0}^{t} \theta^2(s)\,ds\right)^{\frac{1}{2}} \left(\int_{t_0}^{t} \exp(-2cs)\left(\int_{t_0}^{s} \exp(c\tau)\psi(\tau)\,d\tau\right)^2 ds\right)^{\frac{1}{2}}.$$

Define

$$\begin{aligned} I(t) &:= \int_{t_0}^{t} \exp(-2cs)\left(\int_{t_0}^{s} \exp(c\tau)\psi(\tau)\,d\tau\right)^2 ds \\ &\le \frac{1}{c}\int_{t_0}^{t} \exp(-cs)\psi(s)\int_{t_0}^{s} \exp(c\tau)\psi(\tau)\,d\tau ds \\ &\le \frac{1}{c}\left(\int_{t_0}^{t} \psi^2(\tau)\,ds\right)^{\frac{1}{2}} (I(t))^{\frac{1}{2}}. \end{aligned}$$

Therefore

$$\int_{t_0}^{t} \exp(-cs)\theta(s)\int_{t_0}^{s} \exp(c\tau)(\theta(\tau)\beta(\tau))^2\,d\tau ds \le \frac{1}{c}\left(\int_{t_0}^{t} \theta^2(s)\,ds\right)^{\frac{1}{2}} \left(\int_{t_0}^{t} (\theta(s)\beta(s))^4\,ds\right)^{\frac{1}{2}}$$

and similarly

$$\int_{t_0}^{t} \exp(-cs)\theta(s) \int_{t_0}^{s} \exp(c\tau)\theta(\tau)\beta(\tau)\, d\tau ds \leq \frac{1}{c}\left(\int_{t_0}^{t} \theta^2(s)\, ds\right)^{\frac{1}{2}} \left(\int_{t_0}^{t} (\theta(s)\beta(s))^2\, ds\right)^{\frac{1}{2}}.$$

Therefore the result is obtained with $a_1 = (d/c)^2 + V^2(x(t_0))$ and $a_2 = 1/c^2$. $\square$

### 7.5.1 The asymptotic tracking objective

We now give the theorem which establishes for $m = 1$ and general $m$, under the control $\Psi_{1,2,j}(\cdot,\cdot)$, $j = 1$ or $j = 2$, (i) the maximal interval of existence of any solution to the differential inclusion (7.8) is the infinite interval $[t_0, \infty)$, (ii) the adaptive gain $k(\cdot)$ tends to a finite limit, (iii) the error asymptotically tends to zero and (iv) the state $x(\cdot)$ is bounded.

**Theorem 7.5.3** *Let $w : [t_0, \omega) \to \mathbb{R}^{n+m+1}$ be a solution to the differential inclusion (7.8) on its maximal interval of existence, then*

*(i) $\omega = \infty$,*

*(ii) $\lim_{t\to\infty} k(t)$ exists and is finite,*

*(iii) $w(\cdot)$ is bounded.*

*(iv) $e(t) \to 0$ as $t \to \infty$,*

**Proof:** Assumption **(A3.2)** gives the existence of a Liapunov function $V(\cdot)$ that satisfies the properties of Theorem $B$.1.1 in the appendix. From Lemma 4.6.1, the fact that $V(\cdot)$ satisfies a global Lipschitz condition with Lipschitz constant $L$ and assumption **(A3.3)**, we get

$$\begin{aligned} V^+(x(t);\dot{x}(t)) &\leq V^+(x(t); f(t,x(t),0)) + V^o(x(t); f^*(t,x(t),e(t)+r(t))) \\ &\leq -qcV(x(t)) + L\gamma_1(1 + \|x(t)\|^{\delta_1})h^*(e(t)+r(t)), \qquad \text{for almost all } t \in [t_0,\omega). \end{aligned}$$

By the usual Young's inequality argument, using (4.9), letting $r^\dagger := 1/(1-\delta_1)$ and $c^* := (1-\delta_1)[L\gamma_1/(qc/2\delta_1)^{\delta_1}]^{\frac{1}{1-\delta_1}}$, we get for almost all $t \in [t_0, \omega)$,

$$\begin{aligned} V^+(x(t);\dot{x}(t)) + \frac{1}{2}qcV(x(t)) &\leq L\gamma_1 h^*(e(t)+r(t)) + c^* h^{*r^\dagger}(e(t)+r(t)) \\ &\leq L\gamma_1\gamma_6(h^*(e(t)) + h^*(r(t))) + c^*\gamma_6(h^*(e(t)) + h^*(r(t)))^{r^\dagger} \\ &\leq L\gamma_1\gamma_6(h^*(e(t)) + h^*(r(t))) + c^*\gamma_6 c^\dagger(h^{*r^\dagger}(e(t)) + h^{*r^\dagger}(r(t))) \\ &\leq (L\gamma_1\gamma_6\gamma_{2,1} + c^*\gamma_6 c^\dagger\gamma_{2,r^\dagger})\|e(t)\|\rho(e(t)) + c_0, \end{aligned} \tag{7.13}$$

with $c^\dagger$ as in Lemma 7.5.1 and $c_0 := \gamma_6[\gamma_1 L\gamma_{2,1} + c^* c^\dagger \gamma_{2,r^\dagger}]\rho^*$ where $\rho^* := \max\{\|r\|_{1,\infty}\rho(y) \mid y \in \|r\|_{1,\infty}\overline{\mathbb{B}}\}$. Therefore Lemma 7.5.2 holds.

To show the boundedness of $k(\cdot)$, consider the function $E : \mathbb{R}^m \to \bar{\mathbb{R}}_+$, $E : e \mapsto \|e\|^2/2$. For all $(t, w) \in \mathbb{R} \times \mathbb{R}^{n+m+1}$, $e \neq 0$, we get

$$\begin{aligned}\langle e, \eta\rangle &\leq \langle e, g(t,x,y)\rangle + \max_{u\in\Psi_{1,2,2}(t,w)}\{\langle e, H(t,x,y)u\rangle\} + \|r\|_{1,\infty}\|e\| \\ &\leq \gamma_3(1+\|x\|^{\delta_2})\|e\|l(y) + \|r\|_{1,\infty}\|e\| - k\psi^*_{1,2}(t,w)\|e\|^{-1}\langle e, H(t,x,y)K(k)e\rangle \\ &\leq e_1\|e\|[1+\rho(y)]\gamma(t,y) + \|e\|V^2(x) - k\nu(k)\|e\|\psi^*_{1,2}(t,w)\gamma(t,y)\end{aligned}$$

for all $\eta \in F_2(t,w)$, with $\nu(\cdot)$ as in Lemma 3.5.2 and $e_1 := \epsilon^{-1}(\gamma_3\gamma_{4,1} + \gamma_{4,q}q^{-1}[\gamma_3(\delta_2/2)^{\delta_2/2}]^q + \|r\|_{1,\infty})$, where $q = 2/(2-\delta_2)$, using (4.9). Similarly, in the case $m = 1$, for all $\eta \in F_2(t,w)$,

$$e\eta \leq e_1|e|[1+\rho(y)]\gamma(t,y) + |e|V^2(x) + \mathcal{N}(k)|e|\psi^*_{1,2}(t,w)\gamma(t,y)$$

where $\mathcal{N}(\cdot)$ is a continuous Nussbaum function.

So recalling $\dot{k}(t) = \|e(t)\|\psi^*_{1,2}(t,w)\gamma(t,y(t))$,

$$0 \leq E(e(t)) \leq E(e(t_0)) + (e_1 + \epsilon^{-1}(a_1+a_2))(k(t)-k(t_0)) - \int_{k(t_0)}^{k(t)} k\nu(k)\,dk, \tag{7.14}$$

for all $t \in [t_0, \omega)$, using Lemma 7.5.2. Similarly in the case $m = 1$

$$0 \leq E(e(t)) \leq E(e(t_0)) + (e_1 + \epsilon^{-1}(a_1+a_2))(k(t)-k(t_0)) + \int_{k(t_0)}^{k(t)} \mathcal{N}(k)\,dk,$$

for all $t \in [t_0, \omega)$. By an analogous argument as given in Theorem 4.6.3 and Theorem 5.7.1, if $k(\cdot)$ is assumed to be unbounded, a contradiction arises. Therefore $k(\cdot)$ is bounded. This and (7.14) gives the boundedness of $e(\cdot)$.

Now we show that $x(\cdot)$ and therefore $w(\cdot)$ is bounded. From equation (7.13), the fact that $e(\cdot)$ is bounded and $\rho(\cdot)$ is continuous, there exists a constant $K$ such that for all $\xi \in F_1(t,w)$

$$V^+(x(t);\xi) \leq -\frac{1}{2}qcV(x(t)) + K.$$

A similar argument to that in Theorem 4.6.3 gives the boundedness of $x(\cdot)$ and in turn $w(\cdot)$. This implies $\omega = \infty$ by Theorem $A$.3.3 in the appendix.

All that remains to prove is $e(t) \to 0$ as $t \to \infty$. Note that $e \in L^1$ since $\dot{k}(t) \geq \epsilon\|e(t)\|$ and $k(\cdot)$ is bounded. Also $\dot{e} \in L^\infty$ directly from the equation (7.8) and the fact that $w(\cdot)$ is bounded, $\rho(\cdot)$ is continuous and $\gamma(\cdot,\cdot)$ satisfies $\gamma(\mathbb{R}, K)$ is bounded for bounded $K \subset \mathbb{R}^m$. Therefore by Lemma $D$.2.1 in the appendix the result is proved. □

**Example**

We look at a system which satisfies the structural assumptions with $\rho(y) = \exp(\|y\|)$ and $\gamma(t, y) = 1$. Specifically the example we shall consider is the system with $n = 2$ and $m = 2$:

$$\dot{x}(t) = -x(t) + 5\sin(t)\|y(t)\|^2 y(t)$$
$$\dot{y}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + y(t) + u$$

The function we track is $[r_1(t)\ r_2(t)]^T$, generated by (6.16). With initial conditions $x(0) = [2\ -1]$, $y(0) = [0.5\ 0.25]$ and $k(0) = 2$, the results in Figures $(7-8)$, $(7-9)$ and $(7-10)$ are obtained.

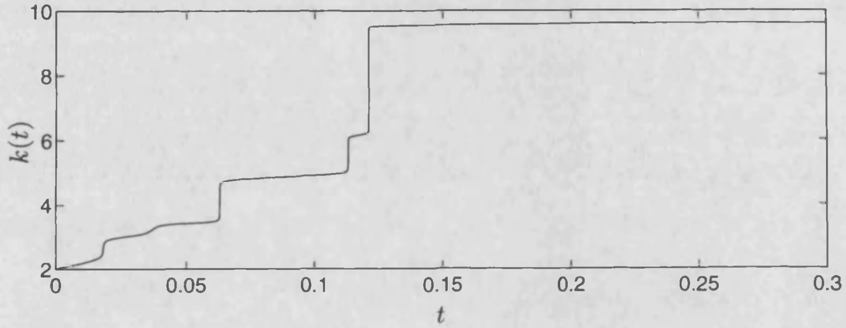


Figure 7-8: The evolution of $k(\cdot)$.

As seen in Figure $(7-8)$, the evolution of $k(\cdot)$ follows the same lines as would be expected. The reader should note that this time the function $k(\cdot)$ increases in steps. At the places where $k(\cdot)$ levels out, the output is 'very close' to the function being tracked.

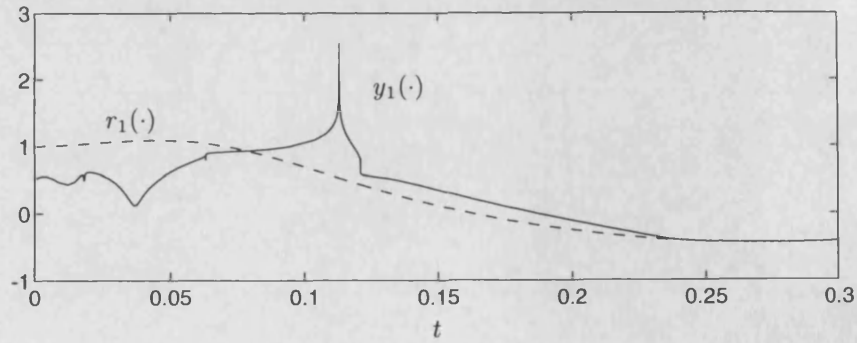


Figure 7-9: The evolution of $y_1(\cdot)$ and $r_1(\cdot)$.

As seen in Figure $(7-9)$, the evolution of $y_1(\cdot)$ tracks the function $r_1(\cdot)$ as $k(\cdot)$ converges.

In Figure $(7-10)$, the evolution of $y_2(\cdot)$ tracks the function $r_2(\cdot)$ as $k(\cdot)$ converges.

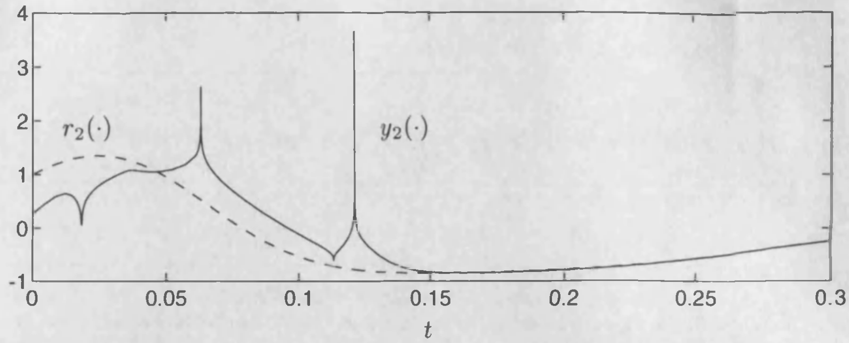


Figure 7-10: The evolution of $y_2(\cdot)$ and $r_2(\cdot)$.

### 7.5.2 The $\lambda$-tracking problem

We now consider the $\lambda$-tracking problem for this subclass. The structural assumptions remain the same, but we use the control $\Psi_{2,2,j}(\cdot,\cdot)$, where $j = 1$ or 2.

From the theory already established we can immediately establish (i) the maximal interval of existence of any solution to the differential inclusion (7.8) is the infinite interval $[t_0, \infty)$, (ii) the adaptive gain $k(\cdot)$ tends to a finite limit, (iii) the state $x(\cdot)$ is bounded, and (iv) the norm of the error asymptotically tends to the interval $[0, \lambda]$.

**Theorem 7.5.4** *Let $w : [t_0, \omega) \to \mathbb{R}^{n+m+1}$ be a solution to the differential inclusion (7.8) on its maximal interval of existence, then*

*(i) $\omega = \infty$,*

*(ii) $\lim_{t\to\infty} k(t)$ exists and is finite,*

*(iii) $w(\cdot)$ is bounded,*

*(iv) $d_\lambda(e(t)) \to 0$ as $t \to \infty$.*

**Proof:** In a similar manner as in the previous Theorem 7.5.3, for the Liapunov function $V(\cdot)$ of Theorem $B$.1.1, for almost all $t \in [t_0, \omega)$

$$V^+(x(t); \dot{x}(t)) + \frac{1}{2} qcV(x(t)) \leq c_1 h^*(y(t)) + c_2 h^{*r^\dagger}(y(t)) \tag{7.15}$$

for some constants $c_1$, $c_2 > 0$ and $r^\dagger \geq 1$. With $V^*(\cdot)$ as defined in (7.4), by writing $y(t) = \nabla V^*(e(t)) + (y(t) - \nabla V^*(e(t))$ and noting $\|y(t) - \nabla V^*(e(t))\| \leq \|r\|_{1,\infty} + \lambda$ we get for almost all $t \in [t_0, \omega)$,

$$V^+(x(t); \dot{x}(t)) + \frac{1}{2} qcV(x(t)) \leq c_1^* d_\lambda(e(t)) \rho(\nabla V^*(e(t))) + c_2^*,$$

for some constants $c_1^*$, $c_2^* > 0$ in a similar manner as Theorem 7.5.3. Therefore Lemma 7.5.2 holds.

As with previous analysis concerning $\lambda$-tracking the boundedness of $k(\cdot)$ is proved by considering the function $V^* : \mathbb{R}^m \to \bar{\mathbb{R}}_+$, as defined in (7.4). For all $(t,w) \in \mathbb{R} \times \mathbb{R}^{n+m+1}$, $e \neq 0$, we get

$$\begin{aligned}\langle \nabla V^*(e), \eta \rangle &\leq d_\lambda(e)\|e\|^{-1}(\langle e, g(t,x,y)\rangle + \langle e, H(t,x,y)u\rangle + \|r\|_{1,\infty}\|e\|) \\ &\leq \gamma_3(1+\|x\|_2^s)d_\lambda(e)l(y) + \|r\|_{1,\infty}d_\lambda(e) \\ &\qquad - k\psi_{2,2}^*(t,w)d_\lambda(e)\|e\|^{-1}\langle e, H(t,x,y)K(k)\psi_\lambda(e)\rangle \\ &\leq e_1 d_\lambda(e)[1+\rho(y)]\gamma(t,y) + d_\lambda(e)V^2(x) - k\nu(k)d_\lambda(e)\psi_{2,2}^*\gamma(t,y)\end{aligned}$$

for all $\eta \in F_2(t,w)$, with $\nu(\cdot)$ as in Lemma 3.5.2 and $e_1$ as in the previous theorem. Similarly, in the case $m=1$, for all $\eta \in F_2(t,w)$

$$\frac{dV^*(e)}{de}\eta \leq e_1 d_\lambda(e)[1+\rho(y)]\gamma(t,y) + d_\lambda(e)V^2(x) + \mathcal{N}(k)d_\lambda(e)\psi_{2,2}(t,w)\gamma(t,y)$$

where $\mathcal{N}(\cdot)$ is a continuous Nussbaum function.

So recalling $\dot{k}(t) = d_\lambda(e(t))\psi_{2,2}^*(t,w(t))\gamma(t,y(t))$

$$0 \leq V^*(e(t)) \leq V^*(e(t_0)) + (e_1 + \epsilon^{-1}(a_1+a_2))(k(t)-k(t_0)) - \int_{k(t_0)}^{k(t)} k\nu(k)\,dk, \tag{7.16}$$

by Lemma 7.5.2. Similarly in the case $m=1$

$$0 \leq V^*(e(t)) \leq V^*(e(t_0)) + (e_1 + \epsilon^{-1}(a_1+a_2))(k(t)-k(t_0)) + \int_{k(t_0)}^{k(t)} \mathcal{N}(k)\,dk.$$

for all $t \in [t_0, \omega)$. By an analogous argument as given in Theorem 4.6.3 and Theorem 5.7.1, if $k(\cdot)$ is assumed to be unbounded, a contradiction arises. Therefore $k(\cdot)$ is bounded. This and (7.16) gives the boundedness of $e(\cdot)$.

The boundedness of $x(\cdot)$ and $\omega = \infty$ can be established by an analogous argument as given in Theorem 7.5.3.

All that remains to prove is $d_\lambda(e(t)) \to 0$ as $t \to \infty$. This is done by using an analogous argument as given in Theorem 7.4.2. □

**Example**

We now consider an example that satisfies the structural assumptions of this section, $n=2$ and $m=2$, with $\rho(y) = \exp(\|y\|)$ and $\gamma(t,y) = 1$. The specific example we consider is:

$$\begin{aligned}\dot{x}(t) &= -x(t) + 5\sin(t)\|y(t)\|^2 y(t) \\ \dot{y}(t) &= 5\sin(10t)\left(\begin{bmatrix}1\\1\end{bmatrix} + y(t)\right) + u.\end{aligned}$$

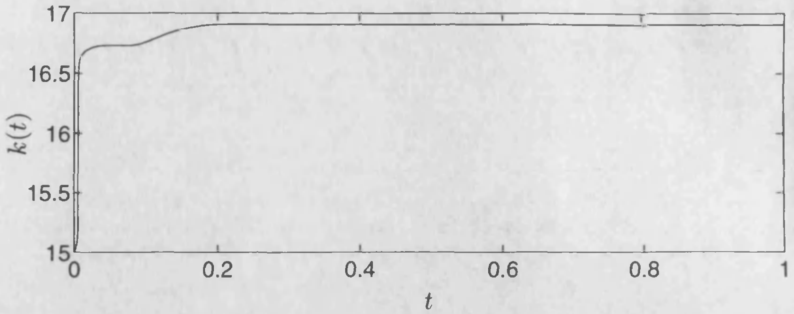


Figure 7-11: The evolution of $k(\cdot)$.

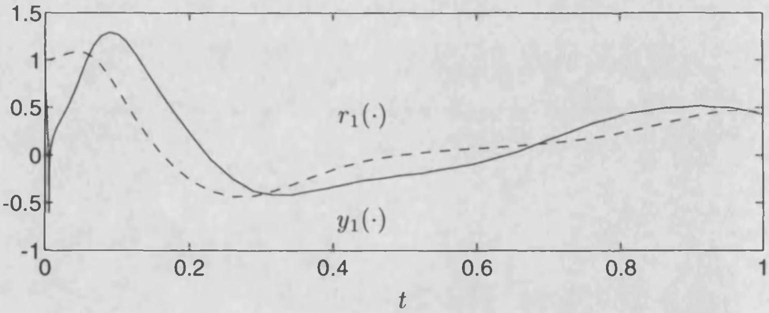


Figure 7-12: The evolution of $y_1(\cdot)$ and $r_1(\cdot)$.

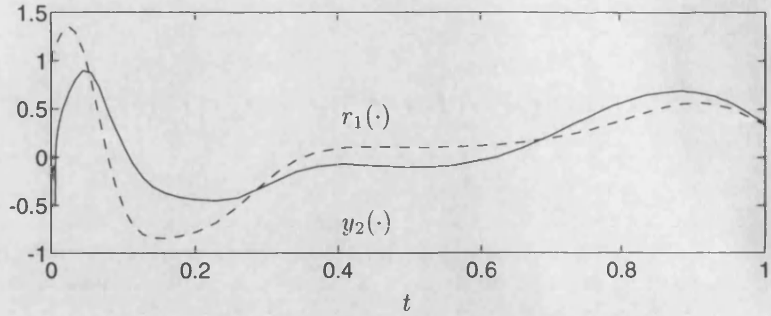


Figure 7-13: The evolution of $y_2(\cdot)$ and $r_2(\cdot)$.

The function the output shall track, to within the distance $\lambda = 0.5$, is the function $[r_1(t)\ r_2(t)]^T$, as generated by (6.16). With initial conditions $x(0) = [2\ -1]^T$, $y(0) = [0.75\ -0.25]^T$ and $k(0) = 15$, the results in Figures $(7-11)$, $(7-12)$ and $(7-13)$, are obtained.

In Figure $(7-11)$ the evolution of $k(\cdot)$ is similar to previous examples.

In Figure $(7-12)$, as $k(\cdot)$ converges $y_1(\cdot)$ tracks the reference signal $r_1(\cdot)$ to within the prescribed distance $\lambda = 0.5$.

In Figure $(7-13)$, as $k(\cdot)$ converges $y_2(\cdot)$ tracks the reference signal $r_2(\cdot)$ to within the prespecified distance $\lambda = 0.5$.

# Chapter 8

# Conclusion

In this thesis we have addressed the problem of achieving control objectives for uncertain nonlinear systems with an adaptive feedback control. The systems studied were linear systems with nonlinear perturbations followed by more general nonlinear systems.

In the case of linear systems with nonlinear perturbations, the uncertainties took the following form: the high-frequency gain matrix was assumed either invertible (for the special case where the input was 2-dimensional) or invertible and essentially column diagonally dominant (for the case where the input was of general dimension). The nonlinear perturbations were known to be bounded, modulo arbitrary scaling, by a known continuous function.

Two different adaptive feedback controls were constructed for the special case where the input was 2-dimensional and the case where the input was of general dimension. In the 2-dimensional case we moved away from the idea of the Mårtensson unmixing set, instead using a strategy which involved a cycling mechanism through the 2-dimensional rotation matrices and a cycling matrix that alternated between positive and negative determinant. In the general $m$-input case, we returned to the idea of Mårtensson unmixing sets, but with the added practicality of a finite set which can be written explicitly for a given dimension.

From these linear systems with nonlinear perturbations, we went on to consider more general nonlinear systems already in a normal form. We studied three subclasses, of this general class of system, characterised by the zero dynamics. A Liapunov function with certain characteristics was assumed to exist for the zero dynamics of the first subclass. We showed, if the right hand side of the state dynamics was a positively homogeneous function, then such a Liapunov function was guaranteed to exist. Originally this subclass was inspired by an example given by Isidori and Byrnes. They showed the equilibrium of the system could not be made attractive with a smooth feedback control. By contrast, the control used in this thesis for this subclass was discontinuous. The equilibrium of the zero dynamics of the second subclass was assumed to be asymptotically stable and the equilibrium of the zero dynamics of the third subclass was assumed to be exponentially stable. The bounds on the other uncertain functions of the system were dependent on the zero dynamics. One uncertainty of note was the input-connection function (in the single-input case) or the input-connection matrix-valued function (in the multi-input case). In the single-input case the function was bounded away from zero and bounded above by a known continuous function of the output and time. In the multi-input case the Gerschgorin discs of the values of the input-connection matrix-valued function $(t, x, y) \mapsto H(t, x, y)$ are contained in the sets $\{a \in \mathbb{C} \,|\, \alpha\gamma(t, y) \leq \mathrm{Re}(a) \leq \beta\gamma(t, y)\}$ and $\{a \in \mathbb{C} \,|\, -\beta\gamma(t, y) \leq \mathrm{Re}(a) \leq -\alpha\gamma(t, y)\}$ for all $(t, x, y) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$ where $\gamma : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}_+$ is a known continuous function and $0 < \alpha \leq \beta$ are positive constants.

Once the control objective of stabilisation was achieved, the control objectives of asymptotic tracking and $\lambda$-tracking were considered. In the first of these objectives, the control forced the output to asymptotically track a given reference signal from the class of reference signals $W^{1,\infty}$. In the second of these objectives, the control forced the output to asymptotically track a given reference signal from this class to within a distance $\lambda$. The latter was achieved with a continuous feedback control. This had the advantage of removing chattering (the high-frequency switching of the control between its variable forms) when the control had a discontinuity.

There are areas where further study is warranted.

- Firstly recall the method of analysis in this thesis did not go through when the tracking objectives were considered in the Isidori and Byrnes inspired subclass, see Chapter 7. Alternative methods of proof could be considered so these control objectives are achieved.

- In Chapters 3, 5, 6 and 7, the unmixing sets for $m \times m$ invertible matrices were such that they could be explicitly written. However, the cost of this practicality was the high-frequency gain or input connection matrix was an element of a subset of the set of invertible matrices. Further research could be done in a return to the structural assumption of the high-frequency gain matrix or the values of the input-connection function being invertible. When considering the matrix-valued functions of Chapters 5 and 7, this task could be far from straightforward, as it is likely an assumption bounding the eigenvalues away from the origin is needed.

- A question not addressed in this thesis is the transient behaviour of the adaptively controlled systems. Some moves have been made in this area by Ilchmann, Logemann and Owens, see [30], [31], [32], [33], [35], [36], [34], [37], [45] and [65], where exponential stability of the closed loop system has been guaranteed.

- All systems in this thesis are high-gain stabilisable. Another area of further research could include considering systems that are only stabilisable for unknown intervals of the gain. S. Townley, [79], has resolved this problem for linear systems.

# Appendix

The appendix shall cover the material that is already well established or is a slight variation of established results. The topics covered in the appendix are the following:

**Appendix A:** Differential Inclusions.
Throughout the thesis differential inclusions play a central role. It has been used for two main reasons. Firstly to aid modelling the uncertainties in the class of system, and secondly to overcome analytical difficulties that arise from discontinuities that occur in the control.

Appendix A begins by giving relevant definitions and properties relating to set-valued maps, such as concepts of upper and lower-semicontinuous set-valued maps and some of their properties. Next we establish the existence of solutions of the initial-value problem

$$\dot{x}(t) \in F(t, x(t)), \qquad x(t_0) = x^0,$$

where $F(\cdot,\cdot)$ is an upper-semicontinuous set-valued map with non-empty, convex and compact values. It should be noted that the theory does not guarantee uniqueness of solution.

Established results are also given concerning maximal solutions and solutions that can be extended to the semi-infinite interval $[t_0, \infty)$. Properties of the $\omega$-limit sets and the asymptotic behaviour of solutions are also investigated.

**Appendix B:** Inverse Liapunov Theory.
Presented with a system which is known to have a stable equilibrium, the question arises as to whether a Liapunov function exists. The answer to this is in the affirmative, given enough structure on the system in question. This idea is the converse of the standard Liapunov Theory.

The Inverse Liapunov theory presented here is essentially the work of Kurzweil [42], Hahn [23] and Yoshizawa [83]. The work in this field has been extensive, from such authors as Massera, [52], in the beginning to Wilson [82] in more recent years.

**Appendix C:** Uniform Distributions.
In chapter 2 of this thesis two methods were given to solve the problem of stabilising a certain class of 2-input systems. (In chapter 6 these ideas were extended to the tracking problem.) One method used the theory of uniform distributions modulo 1. This is the theory that guarantees

the fractional part of a given sequence is uniformly distributed over the interval $[0, 1)$.

**Appendix D:** Miscellaneous Theory.

In the chapters concerned with linear systems with nonlinear perturbations, the integral of cross products terms are encountered. Here we give two important bounds on these types of integral.

We also give a lemma which details the asymptotic behaviour of an absolutely continuous function, $\zeta$ say, such that $\zeta \in L^p$ for $1 \leq p < \infty$ and $\dot{\zeta} \in L^q$ for $1 \leq q \leq \infty$. In the lemma given in this appendix, we find $\zeta(t) \to 0$ in the limit.

# Appendix A

# Set-valued maps and differential inclusions

Throughout the thesis we have used the idea of differential inclusions, and consequently set-valued maps. The main area of use has been in overcoming the analytical problems that could arise in ordinary differential equation theory where the right hand side is discontinuous. In such circumstances the right hand side has been embedded in a set-valued map.

## A.1 Set-valued maps

The following is a description of set-valued maps, and the types of continuity that can be imposed on such maps. For more information on set-valued maps, we refer the reader to [3].

**Notation:** Let $Y$ be a set, then $\mathcal{P}(Y)$ is the power set of $Y$, the set of all subsets of $Y$.

**Definition:** Let $X$ and $Y$ be two sets. A *set-valued map* $F$ from $X$ to $\mathcal{P}(Y)$ is a map such that for all $x \in X$, $F(x) \subset Y$.

**Definition:** The subsets $F(x)$ are called the values of $F$.

**Definition:** The graph of the set-valued map $F$ is the following

$$\mathrm{Graph}(F) := \{(x, y) \in X \times Y \mid y \in F(x)\}.$$

**Definition:** A set-valued map has *convex values* if $F(x)$ is convex for all $x \in X$.

**Definition:** A set-valued map has *compact values* if $F(x)$ is compact for all $x \in X$.

## A.2 Continuity of set-valued maps

Let $X$ and $Y$ be Hausdorff topological spaces. Let $F : X \to \mathcal{P}(Y)$ be a set-valued map with non-empty values.

**Definition:** $F$ is *upper-semicontinuous at* $x_0 \in X$ if for any open $N \subset Y$ containing $F(x_0)$, there exists a neighbourhood $M$ of $x_0$ such that $F(M) \subset N$.

**Definition:** $F$ is *upper-semicontinuous*, if $F$ is upper-semicontinuous at every $x_0 \in X$.

**Definition:** $F$ is *lower-semicontinuous at* $x_0 \in X$ if for any $y_0 \in F(x_0)$ and any neighbourhood $N(y_0)$ of $y_0$, there exists a neighbourhood $N(x_0)$ of $x_0$ such that

$$\forall x \in N(x_0), \qquad F(x) \cap N(y_0) \neq \emptyset.$$

**Definition:** $F$ is *lower-semicontinuous* if it is lower-semicontinuous at every $x_0 \in X$.

**Definition:** $F$ is *continuous* at $x_0 \in X$ if it is both upper-semicontinuous at $x_0$ and lower-semicontinuous at $x_0$.

**Definition:** $F$ is *continuous* if it is both upper-semicontinuous and lower-semicontinuous.

**Proposition A.2.1** *Let $F$ and $G$ be two set-valued maps from $X$ to the subsets of $Y$, and from $Y$ to the subsets of $Z$ respectively. Define $G \circ F$ by*

$$G \circ F(x) := \cup_{y \in F(x)} G(y).$$

*If $F$ and $G$ are upper-semicontinuous set-valued maps, then so is $G \circ F$.*

**Proof:** Let $N$ be open containing $G \circ F(x_0)$. By the upper-semicontinuity of the set-valued map $G$, for all $y \in F(x_0)$, there exists a neighbourhood $M_y$ of $y$ such that $F(M_y) \subset N$. Since $F$ is an upper-semicontinuous set-valued map, given $M_y$, there exists a neighbourhood $O_y$ of $x_0$ such that $F(O_y) \subset M_y$. So the neighbourhood $\cup_y O_y$ satisfies $G \circ F(\cup_y O_y) \subset G(\cup_y M_y) \subset N$. □

**Lemma A.2.2** *If $F_1 : X \to \mathcal{P}(Y)$ and $F_2 : X \to \mathcal{P}(Z)$ are upper-semicontinuous set-valued maps, then the set-valued map $x \mapsto F(x) = F_1(x) \times F_2(x)$ is upper-semicontinuous.*

**Proof:** Let $N = N_1 \times N_2 \subset Y \times Z$ be a neighbourhood of the set $F_1(x) \times F_2(x) = \{(x_1, x_2) \mid x_1 \in F_1(x), x_2 \in F_2(x)\}$. By the upper-semicontinuity of the set-valued map $F_i$ there exists a neighbourhood $M_i$ of $x$ such that $F_i(M_i) \subset N_i$, $i \in \{1, 2\}$. So the neighbourhood $M_1 \cap M_2 \neq \emptyset$ satisfies $F(M_1 \cap M_2) \subset N$. □

**Theorem A.2.3** *Let $F$ be a set-valued map from $X$ to a compact space $Y$ whose graph is closed, then $F$ is upper-semicontinuous.*

**Proof:** See [3, Chapter 1, Corollary 1]. □

So for example the function sgn: $\mathbb{R} \to \{-1, 0, 1\}$ defined as

$$\operatorname{sgn}(x) := \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

can be embedded in the upper-semicontinuous set-valued map

$$\psi(x) = \begin{cases} \{1\}, & x > 0 \\ [-1, 1], & x = 0 \\ \{-1\}, & x < 0 \end{cases}.$$

The next theorem has similarities with single-valued continuous functions.

**Theorem A.2.4** *Let $F$ be an upper-semicontinuous set-valued map with compact values from a compact space $X$ to $\mathcal{P}(Y)$, then $F(X)$ is compact.*

**Proof:** [3, Chapter 1, Proposition 3] □

As the set-valued maps we shall be considering in this thesis will have compact values, we give a more usable and equivalent (in this case) definition of upper and lower-semicontinuous set-valued maps, when $X$ and $Y$ are normed spaces.

**Definition:** Let $X$ and $Y$ be normed spaces, $F : X \to \mathcal{P}(Y)$. $F$ is *upper-semicontinuous at* $x_0$ iff given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$F(x_0 + \delta\mathbb{B}) \subset F(x_0) + \epsilon\mathbb{B}.$$

$F$ is *lower-semicontinuous at* $x_0$ iff given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\forall x \in x_0 + \delta\mathbb{B} \implies F(x_0) \subset F(x) + \epsilon\mathbb{B}.$$

These definitions should not be confused with the definitions of upper, and lower-semicontinuous single-valued functions. However, as these concepts will be needed later, we make clear the difference by giving their definitions.

**Definition:** A single-valued function $f : X \to \mathbb{R}$ is upper-semicontinuous at $x_0$ if given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\|x - x_0\| < \delta \implies f(x) < f(x_0) + \epsilon$$

$f(\cdot)$ is upper-semicontinuous if it is upper-semicontinuous at each $x \in X$.

**Definition:** A single-valued function $f : X \to \mathbb{R}$ is lower-semicontinuous at $x_0$ if given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\|x - x_0\| < \delta \implies f(x_0) < f(x) + \epsilon$$

$f(\cdot)$ is lower-semicontinuous if it is lower-semicontinuous at each $x \in X$.

**Recall:** A single-valued function $f : X \to \mathbb{R}$ is continuous if it is both upper-semicontinuous and lower-semicontinuous.

**Lemma A.2.5** *Let $F : X \to \mathcal{P}(Y)$ and $G : X \to \mathcal{P}(Y)$ be upper-semicontinuous set-valued maps, then $F + G : X \to \mathcal{P}(Y)$, defined as*

$$(F + G)(x) := F(x) + G(x) \subset \mathcal{P}(Y)$$

*is an upper-semicontinuous set-valued map.*

**Proof:** Given $\epsilon > 0$, there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that for all $x \in X$

$$F(x + \delta_1 \mathbb{B}) \subset F(x) + \frac{\epsilon}{2}\mathbb{B} \text{ and } G(x + \delta_2 \mathbb{B}) \subset G(x) + \frac{\epsilon}{2}\mathbb{B}.$$

Define $\delta := \min\{\delta_1, \delta_2\}$. So for $\bar{x} \in x + \delta\mathbb{B}$

$$(F + G)(\bar{x}) = F(\bar{x}) + G(\bar{x}) \subset F(x) + G(x) + \epsilon\mathbb{B}.$$

This concludes the proof. □

The following is an extension to a result given by Goodall, [21].

**Lemma A.2.6** *Let $f : X \to \mathbb{R}^{n\times m}$ be a continuous single-valued function from a normed space $X$ to the set of real $n \times m$ matrices in the sense that*

$$\|x - \bar{x}\| < \delta \implies \|f(x) - f(\bar{x})\| < \epsilon$$

*where the second of these norms is a matrix norm. Let $F : X \to \mathcal{P}(\mathbb{R}^m)$ be an upper-semicontinuous set-valued map with compact values, then $fF$ defined as*

$$(fF)(x) := \{f(x)u \mid u \in F(x)\}$$

*is an upper-semicontinuous map with compact values.*

**Proof:** The values of $fF$ are trivially compact. Let $\bar{x} \in X$. First consider the case $f(\bar{x}) \neq 0$. Let $\epsilon > 0$ be arbitrary. Define $\epsilon_2 := \epsilon/(2\|f(\bar{x})\|)$ and $\epsilon_1 := \epsilon/(2(\max\{\|u\| \mid u \in F(\bar{x})\} + \epsilon_2))$.

There exists a $\delta_1 > 0$ such that

$$\|x - \bar{x}\| < \delta_1 \implies \|f(x) - f(\bar{x})\| < \epsilon_1$$

and there exists a $\delta_2 > 0$ such that

$$F(\bar{x} + \delta_2 \mathbb{B}) \subset F(\bar{x}) + \epsilon_2 \mathbb{B}.$$

Let $\delta := \min\{\delta_1, \delta_2\}$. So for $x \in \bar{x} + \delta\mathbb{B}$, given $u \in F(x)$ there exists a $v \in F(\bar{x})$ such that $\|u - v\| < \epsilon_2$. So write

$$f(x)u = f(\bar{x})v + f(x)u - f(\bar{x})u + f(\bar{x})u - f(\bar{x})v.$$

Note that $f(x)u \in (fF)(x)$ and $f(\bar{x})v \in (fF)(\bar{x})$, and

$$\begin{aligned} \|f(x)u - f(\bar{x})u + f(\bar{x})u - f(\bar{x})v\| &\leq \|f(x)u - f(\bar{x})u\| + \|f(\bar{x})u - f(\bar{x})v\| \\ &< \epsilon_1(\max\{\|u\| \mid u \in F(\bar{x})\} + \epsilon_2) + \|f(\bar{x})\|\epsilon_2 = \epsilon. \end{aligned} \tag{A.1}$$

Therefore $(fF)(x) \subset (fF)(\bar{x}) + \epsilon\mathbb{B}$. This concludes the proof in the case where $f(\bar{x}) \neq 0$.

In the case $f(\bar{x}) = 0$ we must show, given $\epsilon > 0$ there exists a $\delta > 0$ such that $(fF)(\bar{x}+\delta\mathbb{B}) \subset \epsilon\mathbb{B}$. This is done by simply letting $\epsilon_2$ be arbitrary and equating $\epsilon_1 := \epsilon/(\max\{\|u\| \mid u \in F(\bar{x})\} + \epsilon_2)$. □

**Lemma A.2.7** *Let $X$ and $Y$ be normed spaces, $F : X \to \mathcal{P}(Y)$ be a continuous set-valued map with compact values, then the single-valued function $f : X \to \mathbb{R}_+$, defined as*

$$f(x) := \ max\{\|\phi\| \mid \phi \in F(x)\}$$

*is well defined and continuous.*

**Proof:** Since the values of $F$ are compact, the function $f$ is well defined.

The continuity of $f$ is proven by showing it is both an upper, and lower semi-continuous single-valued function. Let $\epsilon > 0$ be arbitrary. By upper-semicontinuity of the set-valued map $F$, there exists a $\delta_1 > 0$ such that

$$\|x - x_0\| < \delta_1 \implies F(x) \subset F(x_0) + \epsilon\mathbb{B}.$$

So for all $\phi_x \in F(x)$, $\|\phi_x\| \leq f(x_0) + \epsilon$, since $F(x)$ is contained in $F(x_0) + \epsilon\mathbb{B}$. Therefore $f(x) \leq f(x_0) + \epsilon$, so upper-semicontinuity is established.

By lower-semicontinuity of the set-valued map $F$, there exists a $\delta_2 > 0$ such that

$$\|x - x_0\| < \delta_2 \implies F(x_0) \subset F(x) + \epsilon\mathbb{B}.$$

So for all $\phi_{x_0} \in F(x_0)$, $\|\phi_{x_0}\| \leq f(x) + \epsilon$. Therefore $f(x_0) \leq f(x) + \epsilon$, so lower-semicontinuity is established. Therefore $f$ is continuous. □

## A.3 Some results concerning differential inclusions

Here we gather some results concerned with differential inclusions: much of this material can be found in, for example [73], [67] and [70]. For completeness we present some of this theory, as well as other results used in this thesis which are in a similar vein.

We shall consider the properties of the solutions of the non-autonomous initial-value differential inclusion problem

$$\begin{aligned} &\dot{x}(t) \in F(t, x(t)), \qquad (t, x(t)) \in \mathbb{R} \times \mathbb{R}^N \\ &x(t_0) = x^0 \in \mathbb{R}^N \end{aligned} \tag{A.2}$$

where $F(\cdot,\cdot)$ is an upper-semicontinuous set-valued map with non-empty, compact and convex values. By a solution we mean a function $t \mapsto x(t) \in \mathbb{R}^N$, defined on some interval $[0, \omega)$, which is absolutely continuous on compact subintervals, and which satisfies the inclusion of $(A.2)$ almost everywhere with the initial condition $x(t_0) = x^0$.

**Theorem A.3.1** *Let $F : \mathbb{R} \times \mathbb{R}^N \to \mathcal{P}(\mathbb{R}^N)$ be upper-semicontinuous with non-empty, convex and compact values. For each $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^N$, the initial-value problem (A.2) has a solution $x : [t_0, \omega) \to \mathbb{R}^N$, $\omega > t_0$.*

**Proof:** See [3, Chapter 2, Theorem 3]. □

**Definition:** Suppose $0 < \omega \leq \infty$ and $x : [t_0, t_0 + \omega) \to \mathbb{R}^N$ is a solution of the initial-value problem $(A.2)$. The interval $[t_0, t_0 + \omega)$ is a *maximal interval of existence*, and the solution $x(\cdot)$ is said to be *maximal*, if $x(\cdot)$ does not have a proper extension which is also a solution of $(A.2)$.

**Theorem A.3.2** *Every solution of (A.2) can be extended into a maximal solution.*

Before we give the proof, we remind the reader of Zorn's Lemma.

**Zorn's Lemma:** Let $\mathcal{A} \neq \emptyset$ be a partially ordered set. If every totally ordered subset $\mathcal{T}$ of $\mathcal{A}$ has an upper bound, then $\mathcal{A}$ has at least one maximal element.

**Proof of Theorem** $A.3.2$**:** With $0 < \tau \leq \infty$, denote $I(\tau)$ as the interval $[t_0, t_0 + \tau)$. Let $x : I(\omega) \to \mathbb{R}^N$ be a solution of $(A.2)$ for some $0 < \omega \leq \infty$. Let

$$\mathcal{A} = \{(\rho, y) \,|\, \omega \leq \rho \leq \infty,\, y : I(\rho) \to \mathbb{R}^N \text{ is a solution of } (A.2),\ y(t) = x(t)\, \forall t \in I(\omega)\}$$

and on this non-empty set $\mathcal{A}$, define a partial ordering $\preceq$ by $(\rho_1, y_1) \preceq (\rho_2, y_2)$ if and only if $\rho_1 \leq \rho_2$ and $y_2(t) = y_1(t)$ for all $t \in I(\rho_1)$.

The theorem is proved if $\mathcal{A}$ has a maximal element: Let $\mathcal{T}$ be any totally ordered subset of $\mathcal{A}$. Let $P = \sup\{\rho \,|\, (\rho, y) \in \mathcal{T}\}$ and let $Y : I(P) \to \mathbb{R}^N$ be defined by the property that, for every $(\rho, y) \in \mathcal{T}$, $Y(t) = y(t)$ for all $t \in I(\rho)$. Then $(P, Y)$ is in $\mathcal{A}$ and is an upper bound for $\mathcal{T}$. By Zorn's Lemma, it follows that $\mathcal{A}$ contains at least one maximal element. □

**Theorem A.3.3** *Suppose $t_0 < \omega \leq \infty$ and that $x : [t_0, t_0 + \omega) \to \mathbb{R}^N$ is a maximal solution of (A.2) with $F(\cdot, \cdot)$ upper-semicontinuous. If $x(\cdot)$ is bounded, then $\omega = \infty$.*

**Proof:** Suppose $\omega < \infty$. If we can show that $x(\cdot)$ is uniformly continuous, then a contradiction arises since uniformly continuous $x(\cdot)$ extends into a continuous function $[t_0, t_0 + \omega] \to \mathbb{R}^N$, and this in turn extends into a solution on the interval $[t_0, t_0 + \omega^*)$ with $\omega^* > \omega$. This would contradict the *maximality* of $x(\cdot)$.

The proof on uniform continuity follows: Since $x(\cdot)$ is bounded, the closure of its trajectory is compact, and by the property on $F$, it follows that $\dot{x}(\cdot)$ is bounded giving the uniform continuity of $x(\cdot)$. □

## A.3.1 Locally Lipschitz functions

Before we proceed further with the theory, the notion of *locally Lipschitz* for functions $\mathbb{R}^N \to \mathbb{R}$ must be made clear.

**Definition:** A function $V : \mathbb{R}^N \to \mathbb{R}$ is locally Lipschitz if, for every compact $K \subset \mathbb{R}^N$, there exists a scalar $k$ such that

$$|V(x) - V(y)| \leq k\|x - y\|, \qquad \forall x, y \in K.$$

Note that locally Lipschitz function are continuous, but not necessarily differentiable everywhere, however locally Lipschitz functions are differentiable almost everywhere by Rademacher's theorem. Therefore we will work with a generalised derivative, the Clarke's generalised derivative.

**Definition:** The Clarke's generalised derivative of $V$ at $x$ in the direction $\phi$ is defined by

$$V^o(x, \phi) := \limsup_{\substack{y \to x \\ h \downarrow 0}} \frac{V(y + h\phi) - V(y)}{h}$$

If $V$ is differentiable at $x(\cdot)$, then this reduces to $V^o(x, \phi) = \langle \nabla V(x), \phi \rangle$.

Another derivative for Lipschitz functions, is the following

**Notation:** Let $V$ be a locally Lipschitz function defined on an open set $U$ of $\mathbb{R}^n$. Define $V^+$ at $x$ in the direction $v$ as

$$V^+(x;v) := \limsup_{h\downarrow 0} \frac{V(x+hv)-V(x)}{h}, \qquad v \in \mathbb{R}^n \tag{A.3}$$

for all $x \in \mathbb{R}^n$.

The following result concerning the Clarke's derivative is stated, but not proved. The reader can find proofs of these results in [67], [70] and [73].

**Lemma A.3.4** *Let $V : \mathbb{R}^N \to \mathbb{R}$ be locally Lipschitz. For each fixed $x \in \mathbb{R}^N$, the function $V^o(x;\cdot)$ is globally Lipschitz on $\mathbb{R}^N$.*

**Proof:** See [67] and [70]. □

**Lemma A.3.5** *If $V : \mathbb{R}^N \to \mathbb{R}$ is locally Lipschitz and $x : [t_0,\infty) \to \mathbb{R}^N$ is absolutely continuous on compact subintervals, then $V \circ x$ is absolutely continuous on compact subintervals and*

$$\frac{dV(x(t))}{dt} = V^+(x(t);\dot{x}(t)), \qquad \textit{for almost all } t.$$

**Proof:** $x$ is absolutely continuous, and therefore $\dot{x}(t)$ exists almost everywhere. Let $N(x) \subset [t_0,\infty)$ denote the set of measure zero on which $\dot{x}(t)$ fails to exist. We claim

$$\limsup_{h\downarrow 0} \frac{V(x(t+h))-V(x(t))}{h} = V^+(x(t);\dot{x}(t)), \qquad \forall t \in [t_0,\infty)\backslash N(x).$$

Since $V(\cdot)$ is locally Lipschitz, for each $t \in [t_0,\infty)\backslash N(x)$, there exists a $\lambda > 0$ (a Lipschitz constant) such that for all $h > 0$ sufficiently small,

$$\begin{aligned} V(x(t+h)) - V(x(t)) &\leq V(x(t)+h\dot{x}(t)) - V(x(t)) + |V(x(t+h)) - V(x(t)+h\dot{x}(t))| \\ &\leq V(x(t)+h\dot{x}(t)) - V(x(t)) + \lambda\|x(t+h) - x(t) - h\dot{x}(t)\|. \end{aligned}$$

Therefore

$$\limsup_{h\downarrow 0} \frac{V(x(t+h))-V(x(t))}{h} \leq V^+(x(t);\dot{x}(t)).$$

A similar argument gives

$$V(x(t)+h\dot{x}(t)) - V(x(t)) \leq V(x(t+h)) - V(x(t)) + \lambda\|x(t+h) - x(t) + h\dot{x}(t)\|.$$

and consequently

$$V^+(x(t);\dot{x}(t)) \leq \limsup_{h\downarrow 0} \frac{V(x(t+h))-V(x(t))}{h}.$$

We now show $V \circ x$ is absolutely continuous on compact subintervals of $[t_0,\infty)$. Let $[\alpha,\beta] \subset [t_0,\infty)$. Since $[\alpha,\beta]$ is compact $K = x([\alpha,\beta])$ is compact. Since $V(\cdot)$ is locally Lipschitz, $V(\cdot)$

is Lipschitz on $K$. Therefore $V \circ x$ is absolutely continuous on $[\alpha, \beta]$, ie. it is a function that is differentiable almost everywhere and is the integral of its derivative $(V \circ x)' \in L^1[\alpha, \beta]$). This concludes the result. □

### A.3.2 $\omega$-limit sets

We now look at the asymptotic behaviour of solutions of $(A.2)$.

**Definition:** Suppose that $x : [t_0, \omega) \to \mathbb{R}^N$ is a maximal solution of $(A.2)$. A point $\bar{x} \in \mathbb{R}^N$ is a *$\omega$-limit point* of $x(\cdot)$ if there exists an increasing sequence $(t_n) \subset [t_0, \omega)$ such that $t_n \to \omega$ and $x(t_n) \to \bar{x}$ as $n \to \infty$. The set $\Omega(x)$ of all $\omega$-limit points of $x(\cdot)$ is the $\omega$-limit set of $x(\cdot)$.

**Definition:** Let $C \subset \mathbb{R}^N$ be non-empty. A maximal solution $x : [t_0, \omega) \to \mathbb{R}^N$ of $(A.2)$ is said to approach $C$ if $d_C(x(t)) \to 0$ as $t \to \omega$, where $d_C$ is the (Euclidean) distance function for $C$ defined on $\mathbb{R}^N$ by

$$d_C(v) := \inf\{\|v - c\| \mid c \in C\}. \tag{A.4}$$

**Theorem A.3.6** *If $x(\cdot)$ is a bounded maximal solution of (A.2), then $\Omega(x) \subset \mathbb{R}^N$ is non-empty, compact, connected and is the smallest closed set that $x(\cdot)$ approaches.*

**Proof:** The boundedness of $x(\cdot)$ implies that $\Omega(x)$ is non-empty and compact subset of $\mathbb{R}^N$, since $\Omega(x) = \cap_{t_0 \le t \le \omega} cl(x([t, \omega)))$.

By Theorem $A.3.3$, $x(\cdot)$ has maximal interval of existence $[t_0, \infty)$. Suppose that $x(\cdot)$ does not approach $\Omega(x)$. Then since $x(\cdot)$ is bounded and $d_{\Omega(x)}(\cdot)$ is continuous, there exists an increasing sequence $(t_n)$ with $t_n \to \infty$ such that $(x(t_n))$ converges to $\bar{x}$ say, and $d_{\Omega(x)}(\bar{x})$ is non-zero. Clearly this is a contradiction, since by definition $\bar{x} \in \Omega(x)$. So $x(t)$ approaches $\Omega(x)$. $\Omega(x)$ is the smallest closed subset approached by $x(t)$ follows from the observation: if $C$ is closed and $x(t)$ approaches $C$, then $\Omega(x) \subset C$.

Now suppose for contradiction $\Omega(x)$ is not connected. Then the compact set $\Omega(x)$ is the disjoint union of non-empty compact sets $\Omega_1$ and $\Omega_2$. Since $x(\cdot)$ approaches $\Omega(x)$, it follows that $x(t)$ either approaches $\Omega_1$ or $x(t)$ approaches $\Omega_2$. This contradicts the fact that $\Omega(x)$ is the smallest closed set that $x(t)$ approaches. Therefore $\Omega(x)$ is connected. □

**Theorem A.3.7** *Let $V : \mathbb{R}^N \to \mathbb{R}$ be locally Lipschitz. Let $F(\cdot,\cdot)$ of (A.2) be such that $F(\mathbb{R}, K)$ is bounded for bounded $K \subset \mathbb{R}^n$. Suppose $q : \mathbb{R}^N \to \mathbb{R}$ is continuous and for all $(t, z) \in \mathbb{R} \times \mathbb{R}^N$*

$$V^o(z; \phi) \le q(z) \le 0, \qquad \forall \phi \in F(t, z).$$

*If $x(\cdot)$ is a bounded solution of (A.2) then $x(\cdot)$ approaches the set*

$$\Sigma := \{z \in \mathbb{R}^N \mid q(z) = 0\}.$$

**Proof:** Since $x(\cdot)$ is bounded, it has an interval of existence of $[t_0, \infty)$ by Theorem $A.3.3$. Also by Lemma $A.3.5$, $V \circ x$ is absolutely continuous on compact subintervals.

By hypothesis, $q(z) \le 0$ for all $z \in \mathbb{R}^N$ and so we have

$$V(x(t)) - V(x(t_0)) \le \int_{t_0}^{t} q(x(s))\, ds \le 0, \qquad t_0,\, t \in \mathbb{R}_+,\, t_0 \le t, \tag{A.5}$$

and so $V(x(\cdot))$ is monotone decreasing.

We can now show that $\Sigma$ is non-empty and is approach by $x$. Since, by Theorem $A.3.6$, bounded $x(\cdot)$ approaches its non-empty $\omega$-limit set $\Omega(x)$. Suppose for a contradiction there exists an $\bar{x} \in \Omega(x)$ such that $\bar{x} \notin \Sigma$. So $q(\bar{x}) \le -2\epsilon$ for some $\epsilon > 0$. By the continuity of $q(\cdot)$, for some $\delta > 0$

$$\|x - \bar{x}\| < \delta \implies q(x) < -\epsilon.$$

Since $\bar{x} \in \Omega(x)$, there exists an increasing sequence $(t_n)$ with $t_n \to \infty$ and $x(t_n) \to \bar{x}$. So there exists an $N_1 \in \mathrm{N}$ such that $q(x(t_n)) < -\epsilon$ for all $n > N_1$.

By the property on $F$ and the boundedness of $x(\cdot)$, $F(\mathbb{R}, cl(x([t_0, \infty)))) \subset r\mathbb{B}$, for some positive $r$. By continuity of $V$ we have for some $N_2 \ge N_1$,

$$V(x(t_n)) - V(\bar{x}) < \frac{\delta\epsilon}{4r} \tag{A.6}$$

for all $n > N_2$.

Let $N > N_2$ be such that $x(t_n) \in \bar{x} + \frac{1}{2}\delta\mathbb{B}$ for all $n > N$. Hence, using the monotonicity of $V(x(\cdot))$ and $(A.5)$, we conclude

$$V(x(t_n)) - V(\bar{x}) \ge V(x(t_n)) - V(x(t_n + (\delta/3r))) \ge -\int_{t_n}^{t_n+(\delta/3r)} q(x(s))\, ds \ge \frac{\delta\epsilon}{3r}$$

for all $n > N$. This contradicts $(A.6)$. Thus $q(\bar{x}) = 0$ for all $\bar{x} \in \Omega(x)$ and so $\Omega(x) \subset \Sigma$. □

**Theorem A.3.8** *Let $V$ be a locally Lipschitz, positive-definite function. Let $c \in Image(V)$ and define $V^* := \cup_{0 \le r \le c} V^{-1}(r)$. Suppose there exists an $m > 0$ such that for all $(t, x) \in \mathbb{R} \times \mathbb{R}^N$ with $x \notin V^*$*

$$V^+(x; \phi) \le -m, \qquad \forall \phi \in F(t, x). \tag{A.7}$$

*If $x(\cdot) : [t_0, \infty) \to \mathbb{R}^N$ is a bounded solution of (A.2) then $x(t) \to V^*$ as $t \to \infty$.*

**Proof:** Since $x(\cdot)$ is bounded, it has an interval of existence of $[t_0, \infty)$ by Theorem $A.3.3$.

By the continuity of $V$, $V^*$ is closed. If there exists a $\tau \in [t_0, \infty)$ such that $x(\tau) \in V^*$, then $x(t) \in V^*$ for all $t \geq \tau$ by the following argument: assume for a contradiction that $t > \tau$ with $x(t) \notin V^*$. Then there exists $t^* \in [\tau, t)$ such that $x(s) \notin V^*$ for $s \in (t^*, t]$, $x(t^*) \in V^*$ and

$$V(x(t)) - V(x(t^*)) = \int_{t^*}^{t} (V \circ x)'(s)\, ds \leq -m(t - t^*) < 0.$$

This is a contradiction, as $V(x(t)) \geq V(x(t^*))$ by the construction of $V^*$.

Now suppose that $x(t) \notin V^*$ for all $t \in [t_0, \infty)$ then two possibilities occur: (i) $\Omega(x) \subset V^*$, or (ii) $\Omega(x) \not\subset V^*$. If (i) is the case there is nothing left to prove. So suppose for a contradiction that (ii) is the case. Let $\bar{x} \in \Omega(x)$ be such that $\bar{x} \notin V^*$. So there exist $(t_n)$ with $t_n \to \infty$ and $x(t_n) \to \bar{x}$. Since $V^*$ is closed, there exists an $N \in \mathrm{N}$ such that $x(t_n) \notin V^*$ for $n \geq N$. Given $\epsilon > 0$ sufficiently small, since $t \mapsto V(x(t))$ is decreasing outside $V^*$, for all $n$

$$V(x(t_n)) - V(\bar{x}) \geq V(x(t_n)) - V(x(t_n + (\epsilon/m))) \geq + \int_{t_n}^{t_n + (\epsilon/m)} m\, ds = \epsilon.$$

But by the continuity of $V$ we have

$$V(x(t_n)) - V(\bar{x}) < \epsilon$$

for all $n$ sufficiently large which gives a contradiction. Therefore (ii) is not possible, which concludes the proof. □

# Appendix B

# Inverse Liapunov theory

Inverse Liapunov theory addresses the question: given a system

$$\dot{x}(t) = f(t, x(t)), \qquad x(t_0) \in \mathbb{R}^n \tag{B.1}$$

with a (uniformly asymptotically) stable equilibrium does a Liapunov function exist? This question has been addressed by many authors, a history is given in Kurzweil's paper [42], and is briefly given here. This question was first answered by mathematicians in Kazan. Persidskii in [66] found, if an equilibrium of ($B$.1) is stable and $f(t, \cdot)$ is $C^1$, then there exists a function $(x, t) \mapsto V(x, t)$ that satisfies the conditions of Liapunov's first theorem. In [43], Kurzweil gives the same result when an equilibrium of ($B$.1) is uniformly stable.

J. L. Massera was the first to study the inversion of Liapunov's second theorem in [52]. Here he proved that if an equilibrium of ($B$.1) is asymptotically stable and if the function $f(t, x)$ is periodic in $t$ for fixed $x$, and the derivatives $\frac{\partial}{\partial x_j} f_i$ are continuous, then there exists a function $V(x, t)$ which satisfies the conditions of Liapunov's second theorem. (He also extended the theory so that if $f(t, x)$ is independent of the variable $t$, then $V$ is a function of just $x$.) I. G. Malkin, in [47], made a further extension to the theory by giving an inverse theorem when the flow of ($B$.1), $\phi(t + t_0, t_0, x^0)$ say, tends to zero as $t \to \infty$ uniformly and when the partial derivatives $\frac{\partial}{\partial x_j} f_i$ are continuous and bounded. E. A. Barbasin and N. N. Krasovskii in [4] give a similar theorem for the case where an equilibrium is asymptotically stable in the large, ie. the flow $\phi(t, t_0, x^0) \to 0$ for every $x^0$ and $t_0 \geq 0$. V. I. Zubov has investigated other types of stability in his papers [86] and [87].

Other work on Inverse Liapunov theory can be found in Hahn [23]. Results on exponentially stable systems as well as asymptotically stable systems can be found in this reference. The result by Yoshizawa, [83], given in this appendix of this thesis, concerning exponentially stable systems has weaker conditions imposed on the system.

F. Wesley Wilson, Jr gives a very succinct proof of Inverse Liapunov theory for asymptotically stable systems in his paper [82], see also Z. Artstein [1].

The theorems we give in this appendix are less general versions of Kurzweil's, [42], concerning asymptotically stable systems, and Yoshizawa's, [83], concerning exponentially stable systems. These theorems are given, as the assumptions imposed on the systems are minimal in the case of Kurzweil's, and the properties of the Liapunov function are useful to the setting of this thesis in the case of Yoshizawa's. An Inverse Liapunov theorem by Hahn is given, where the right hand side, $f$, is a positively homogeneous function of order $k > 1$.

When we use the inverse Liapunov theory in the analysis of the systems of chapters 4, 5 and 7, we shall equate the function $f$ of this appendix with the zero dynamics function $\bar{f}$ of these chapters. The reader should note that in this appendix we specialise to the autonomous case

$$\dot{x}(t) = f(x(t)), \qquad x(t_0) = x^0 \in \mathbb{R}^n, \tag{B.2}$$

where $f$ is locally Lipschitz (the latter guaranteeing the uniqueness of the solution). Throughout this topic we shall denote the flow of $(B.2)$ as $\phi$, ie. $t \mapsto x(t) = \phi(t-t_0, x^0)$ is the unique solution of $(B.2)$.

**Definition:** The origin is stable if for every $\epsilon > 0$ there exists $\delta > 0$ such that every solution of the equation $(B.2)$ with $\|x(t_0)\| \leq \delta$ satisfies $\|x(t)\| < \epsilon$ for all $t \geq t_0$. In addition, if there exists a $\delta_0 > 0$ such that every solution of the equation $(B.2)$ with $\|x(t_0)\| < \delta_0$ satisfies $x(t) \to 0$ as $t \to \infty$, then the origin is asymptotically stable.

## B.1 Exponentially stable systems

In this subsection we look at Inverse Liapunov theory for exponentially stable systems.

Using the derivative $(A.3)$ the following inverse Liapunov theorem for exponentially stable systems can be given. This is the autonomous and global case of [83, Theorem 19.2, Chapter 5]. It is repeated here, as it is an informative demonstration of the flavour of proofs in this area.

**Theorem B.1.1** *Suppose that $f(\cdot)$ of system (B.2) satisfies $f(0) = 0$, $f(\cdot)$ is globally Lipschitz and the origin is globally exponentially stable in the sense there exists a $c > 0$ and a $K > 0$ such that for all $x^0 \in \mathbb{R}^n$*

$$\|\phi(t, x^0)\| \leq K \exp(-c(t-t_0))\|x^0\|, \qquad \text{for all } t \geq 0,$$

*then there exists a Liapunov function $V(\cdot)$ defined on $\mathbb{R}^n$ which satisfies the following conditions:*

1. $\|x\| \leq V(x) \leq K\|x\|, \qquad \forall x \in \mathbb{R}^n,$

*2.* $|V(x) - V(x^*)| \leq L\|x - x^*\|$ *for all* $x$, $x^* \in \mathbb{R}^n$, *for* $L > 0$,

*3.* $V^+(x; f(x)) \leq -qcV(x)$ *for all* $x \in \mathbb{R}^n$, *where* $0 < q < 1$.

**Proof:** For a constant $q$ such that $0 < q < 1$, define

$$V(x) := \sup_{\tau \geq 0} \|\phi(\tau, x)\| \exp(qc\tau) \tag{B.3}$$

Clearly $\|x\| \leq V(x)$, and for all $x \in \mathbb{R}^n$,

$$\|x\| \leq V(x) \leq \sup_{\tau \geq 0} K \exp(-[1 - q]c\tau)\|x\| \leq K\|x\|, \tag{B.4}$$

which proves 1. (Note that we have used only the global exponential stability of the origin so far.)

We now prove 2. Let $M$ be the global Lipschitz constant for $f$. Let $T$ be such that $K = \exp([1 - q]cT)$. If $\tau \geq T$, then $K \exp(-[1 - q]c\tau)\|x\| \leq \|x\|$. From this and $(B.4)$ it follows $V(x)$ must be defined for $\tau$ such that $0 \leq \tau \leq T$. Therefore for $x$, $\bar{x} \in \mathbb{R}^n$,

$$\begin{aligned}
|V(x) - V(\bar{x})| &= \left| \sup_{0 \leq \tau \leq T} \|\phi(\tau, x)\| \exp(qc\tau) - \sup_{0 \leq \tau \leq T} \|\phi(\tau, \bar{x})\| \exp(qc\tau) \right| \\
&\leq \sup_{0 \leq \tau \leq T} \|\phi(\tau, x) - \phi(\tau, \bar{x})\| \exp(qc\tau) \\
&\leq \exp(qcT) \exp(MT)\|x - \bar{x}\|
\end{aligned}$$

using Gronwall's inequality, see [17]. Letting the constant $L = \exp(qcT)\exp(MT)$ completes the proof of 2.

It remains to prove 3. Let $\bar{x} = \phi(h, x)$, where $h > 0$, then

$$\begin{aligned}
V(\bar{x}) &= \sup_{\tau \geq 0} \|\phi(\tau, \bar{x})\| \exp(qc\tau) \\
&= \sup_{\tau \geq h} \|\phi(\tau, x)\| \exp(qc\tau) \exp(-qch) \\
&\leq V(x) \exp(-qch),
\end{aligned}$$

which implies

$$\frac{V(\bar{x}) - V(x)}{h} \leq V(x) \frac{\exp(-qch) - 1}{h}.$$

Taking the limit as $h$ tends down to zero gives the result. □

This theorem is used in Chapters 4, 5 and 7, where the zero dynamics form an exponentially stable system. Property (3) in the theorem says the value of $V$ decreases exponentially along solutions of $\dot{x}(t) = f(x(t))$. This property is a key feature in the analysis of that category.

## B.2 Positively-homogeneous right hand sides

In this section we look at systems of the form ($B$.2), where the function $f(\cdot)$ is a positively-homogeneous function of some order greater than 1. For such systems with this property we shall construct an Inverse Liapunov function with useful properties.

**Definition:** The function $f : \mathbb{R}^n \to \mathbb{R}^m$ is positively-homogeneous of order $k > 1$ if $f(\cdot)$ satisfies

$$f(cx) = c^k f(x), \qquad \forall x \in \mathbb{R}^n$$

for all $c \geq 0$.

In the setting of this thesis we shall consider the case $k > 1$. We initially give some results for homogeneous functions.

**Lemma B.2.1** *If $f(\cdot)$ is a continuous, positively-homogeneous function of order $k$, then there exists $K_1 > 0$ and $K_2 > 0$ such that*

$$K_1\|x\|^k \leq \|f(x)\| \leq K_2\|x\|^k.$$

**Proof:** For $x = 0$, the result is trivial. Let $K_2 := \max\{\|f(x)\| \mid \|x\| = 1\}$ and $K_1 := \min\{\|f(x)\| \mid \|x\| = 1\} > 0$. For $x \neq 0$,

$$K_1 \leq \|f(\|x\|^{-1}x)\| \leq K_2$$

and $f(\|x\|^{-1}x) = \|x\|^{-k}f(x)$, which gives the result. □

**Lemma B.2.2** *If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a $C^1$, positively-homogeneous function of order $k$, then there exists $K^* > 0$ such that*

$$\|Df(x)\| \leq K^*\|x\|^{k-1}.$$

*where $Df(x) := \left[\frac{\partial f_i(x)}{\partial x_j}\right]_{m\times n}$ is the Jacobian of $f(\cdot)$ evaluated at $x$.*

**Proof:** This is simply proved by noting the functions $\frac{\partial f_i(\cdot)}{\partial x_j}$ (ie. the elements of $Df(\cdot)$) are homogeneous of order $k-1$ by the following argument: For $c \neq 0$

$$\begin{aligned}\frac{\partial f_i(cx)}{\partial x_j} &= \lim_{h\to 0}\frac{f_i(cx + (0,\ldots,0,h,0,\ldots,0)) - f_i(cx)}{h} \\ &= \lim_{c^{-1}h\to 0} c^{k-1}\frac{f_i(x + (0,\ldots,0,c^{-1}h,0,\ldots,0)) - f_i(x)}{c^{-1}h} \\ &= c^{k-1}\frac{\partial f_i(x)}{\partial x_j}.\end{aligned}$$

The case $c = 0$ is a consequence of the above, and the fact that $f(\cdot)$ is $C^1$. Therefore $Df(\cdot)$ is homogeneous of order $k-1$. Lemma $B$.2.1 above completes the result. □

**Lemma B.2.3** *Let $\rho > 0$. Let $x(t)$ be the solution of (B.2) with initial condition $x(\rho^{k-1}t_0) = x^0$, where $f(\cdot)$ is positively-homogeneous of order $k > 1$. Then $\bar{x}(t) := \rho x(\rho^{k-1}t)$ is the solution of (B.2) with initial condition $\bar{x}(t_0) = \rho x^0$.*

**Proof:** Clearly $\bar{x}(t_0) = \rho x(\rho^{k-1}t_0) = \rho x^0$. Also for almost all $t$,

$$\begin{aligned}\dot{\bar{x}}(t) &= \rho\frac{d(\rho^{k-1}t)}{dt}\,\dot{x}(u)|_{u=\rho^{k-1}t}\\ &= \rho^k f(x(\rho^{k-1}t)) = f(\rho x(\rho^{k-1}t))\\ &= f(\bar{x}(t))\end{aligned}$$

which concludes the result. □

In fact this lemma states $\phi(t,t_0,\rho x^0) = \rho\phi(\rho^{k-1}t,\rho^{k-1}t_0,x^0)$ for all $t \geq t_0$.

As in [23, Section 57] we make the transformation ($x \neq 0$) $x = ry$, $r = \|x\|$. For clarity we give the following notation, with the dependencies clearly stated:

$$\begin{aligned}&r(t,t_0,x^0) := \|\phi(t,t_0,x^0)\|\\ &y(t,t_0,x^0) := \phi(t,t_0,x^0)/r(t,t_0,x^0), \qquad x^0 \neq 0\\ &g(t,t_0,x^0) := \langle y(t,t_0,x^0), f(y(t,t_0,x^0))\rangle\end{aligned}$$

We get, for fixed $(t_0,x^0)$,

$$\begin{aligned}&\dot{r}(t,t_0,x^0) = g(t,t_0,x^0)r^k(t,t_0,x^0),\\ &\dot{y}(t,t_0,x^0) = r^{k-1}(t,t_0,x^0)[f(y(t,t_0,x^0)) - g(t,t_0,x^0)y(t,t_0,x^0)], \qquad \|y(t,t_0,x^0)\| = 1.\end{aligned}$$

Integration gives

$$r(t,t_0,x^0) = \left[\|x^0\|^{1-k} + (1-k)s(t,t_0,x^0)\right]^{-\frac{1}{k-1}}, \qquad \forall t \geq t_0. \tag{B.5}$$

where

$$s(t,t_0,x^0) := \int_{t_0}^{t} g(\tau,t_0,x^0)\,d\tau, \qquad \forall t \geq t_0.$$

Hahn gives the following definition of asymptotic stability of the origin of $(B.2)$ (shown in [23] to be equivalent to our earlier definition).

**Definition:** Let $x^0 \in \mathbb{R}^n$. The origin is asymptotically stable, if there exists $\psi(\cdot)$ a continuous strictly increasing function with $\psi(0) = 0$ and $\sigma(\cdot)$ a continuous, strictly decreasing function, with $\sigma(t) \to 0$ as $t \to \infty$, such that if $\phi(t,t_0,x^0)$ is the state at time $t$ starting at $x^0$ at time $t_0$, then $\|\phi(t,t_0,x^0)\| \leq \psi(\|x_0\|)\sigma(t-t_0)$.

We note that asymptotic stability of the origin implies

$$|s(t,t_0,x^0)| \geq c(t-t_0), \qquad \forall t \geq t_0$$

for some constant $c > 0$, independent of $x^0$, by the following argument: for all $t_0 \in \mathbb{R}$ and all $x^0 \in \mathbb{R}^n$ with $\|x^0\| \leq 1$, we have

$$r(t, t_0, x^0) \leq [1 - (k-1)s(t, t_0, x^0)]^{-\frac{1}{k-1}} \leq \psi(1)\sigma(t - t_0), \qquad \forall t \geq t_0.$$

This implies, for all $t_0 \in \mathbb{R}$ and all $x^0 \in \mathbb{R}^n$ with $\|x^0\| \leq 1$,

$$-s(t, t_0, x^0) \geq \frac{\psi^{1-k}(1)\sigma^{1-k}(t - t_0) - 1}{k - 1} =: \sigma^*(t - t_0), \qquad \forall t \geq t_0. \tag{B.6}$$

By stability, $\sigma^*(\cdot)$ is monotone increasing with $\sigma^*(t) \to \infty$ as $t \to \infty$. Let $T > 0$ be such that $\sigma^*(T) > 0$. Therefore $\|\phi(t_0 + T, t_0, x^0)\| \leq \psi(1)\sigma(T) < 1$. By $(B.6)$ $-s(t_0 + T, t_0, x^0) \geq \sigma^*(T) > 0$. Assume $-s(t_0 + nT, t_0, x^0) \geq n\sigma^*(T)$, then

$$\begin{aligned} -s(t_0 + (n+1)T, t_0, x^0) &= -s(t_0 + nT, t_0, x^0) - \int_{t_0+nT}^{t_0+(n+1)T} g(s, t_0, x^0)\, ds \\ &= -s(t_0 + nT, t_0, x^0) \\ &\quad - \int_{t_0+nT}^{t_0+(n+1)T} g(s, t_0 + nT, \phi(t_0 + nT, t_0, x^0))\, ds \\ &\geq n\sigma^*(T) + \sigma^*(T) = (n+1)\sigma^*(T) \end{aligned}$$

Therefore, by induction $-s(t_0 + nT, t_0, x^0) \geq (n+1)\sigma^*(T)$ for all $t_0 \in \mathbb{R}$ and $x^0 \in \mathbb{R}^n$ such that $\|x^0\| \leq 1$. Therefore, for all $t_0 \in \mathbb{R}$ and all $x^0 \in \mathbb{R}^n$ with $\|x^0\| \leq 1$,

$$-s(t, t_0, x^0) \geq \frac{1}{2}\sigma^*(T)(t - t_0), \qquad \forall t \geq t_0 + T,$$

and consequently

$$r(t, t_0, x^0) \leq \left[1 + \frac{1}{2}(k-1)\sigma^*(T)(t - t_0)\right]^{-\frac{1}{k-1}}, \qquad \forall t \geq t_0 + T$$

Note that for $\|x^0\| \leq 1$ and $t \in [t_0, t_0 + T]$,

$$r(t, t_0, x^0) \leq \psi(\|x^0\|)\sigma(t - t_0) \leq \psi(1)\sigma(0) \leq \psi(1)\sigma(0) \left[\frac{1 + \frac{1}{2}(k-1)\sigma^*(T)(t - t_0)}{1 + \frac{1}{2}(k-1)\sigma^*(T)T}\right]^{-\frac{1}{k-1}}.$$

Defining $c^* := \psi(1)\sigma(0)[1 + \frac{1}{2}(k-1)\sigma^*(T)T]^{\frac{1}{k-1}} \geq 1$ (since $\psi(1)\sigma(0) \geq \|\phi(t_0, t_0, x^0)\| = \|x^0\| = 1$ for $\|x^0\| = 1$) we conclude for $x^0$ such that $\|x^0\| = 1$,

$$r(t, t_0, x^0) \leq c^* \left[1 + \frac{1}{2}(k-1)\sigma^*(T)(t - t_0)\right]^{-\frac{1}{k-1}}, \qquad \forall t \geq t_0.$$

But for general $x^0 \in \mathbb{R}^n$ with $\|x^0\| = \rho > 0$

$$\begin{aligned} r(t, t_0, x^0) &= \rho r(\rho^{k-1}t, \rho^{k-1}t_0, x^0/\rho) \\ &\leq \rho c^* \left[1 + \frac{1}{2}(k-1)\sigma^*(T)\rho^{k-1}(t - t_0)\right]^{-\frac{1}{k-1}} \\ &= \left[a\rho^{1-k} + c_1(t - t_0)\right]^{-\frac{1}{k-1}}, \qquad \forall t \geq t_0, \end{aligned} \tag{B.7}$$

where $a$ and $c_1$ are independent of $x^0$.

Also by the continuity of $f(\cdot)$,

$$g(t, t_0, x^0) \leq \|y(t, t_0, x^0)\| \|f(y(t, t_0, x^0))\| \leq K,$$

for some constant $K > 0$ independent of $x^0$. This implies $|s(t, t_0, x^0)| \leq K(t - t_0)$ for all $t \geq t_0$, which gives the inequality

$$r(t + t_0, t_0, x^0) \geq (r_0^{1-k} + c_2(t - t_0))^{-\frac{1}{k-1}}, \qquad \forall t \geq t_0, \tag{B.8}$$

for some constant $c_2 > 0$ independent of $r_0$.

Before we give the main proof of this section, we remind the reader of the Lebesgue Dominated Convergence Theorem:

**Theorem B.2.4** *If $u_n(\cdot)$ are integrable on $E$ (a measurable set) for all $n \in \mathbb{N}$, with $\|u_n(t)\| \leq g(t)$, where $g(\cdot)$ is a real-valued, integrable function on $E$ and $u_n(t)$ converges to $u(t)$ for almost all $t \in \mathbb{R}$, then $u(\cdot)$ is integrable and*

$$\int_E u(t)\, dt = \lim_{n \to \infty} \int_E u_n(t)\, dt.$$

**Proof:** See [17]. □

Using $(B.7)$ and $(B.8)$ we can go on to prove an Inverse Liapunov theorem, given in Hahn's book [23, Section 57].

**Theorem B.2.5** *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be $C^1$ and positively-homogeneous of order $k > 1$. If the origin of the system (B.2) is asymptotically stable, then there exists $V : \mathbb{R}^n \to \mathbb{R}_+$ such that for $m$ suitably large*

- *$V \in C^1$,*
- *there exist constants $a_1$, $a_2 > 0$ such that*

$$a_1 \|x\|^{(m-1)(k-1)} < V(x) < a_2 \|x\|^{(m-1)(k-1)},$$

- *$V$ satisfies* $\langle \nabla V(x), f(x) \rangle = -\|x\|^{m(k-1)}$ ,
- *for a constant* $a_3 > 0$
$$\|\nabla V(x)\| < a_3 \|x\|^{(m-1)(k-1)-1}.$$

**Proof:** Define
$$V(x) := \int_0^\infty \|\phi(\tau, x)\|^{m(k-1)}\, d\tau$$
where $\phi$ is the flow generated by($B.2$) and $m$ suitably large (see below). Clearly $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$. Also, using ($B.7$),
$$V(x) \leq \int_0^\infty \frac{1}{(a\|x\|^{1-k} + c_1\tau)^m}\, d\tau.$$
With the change of variable $u = a\|x\|^{1-k} + c_1\tau$, we get
$$V(x) \leq a_2 \|x\|^{(m-1)(k-1)},$$
where $a_2 = a^{1-m}/[c_1(m-1)]$. Similarly, using ($B.8$) we get
$$V(x) \geq a_1 \|x\|^{(m-1)(k-1)},$$
where $a_1 = 1/[c_2(m-1)]$. This completes the proof of the second part.

The third part is proved by observing
$$\begin{aligned} V(\phi(t, x^0)) &= \int_0^\infty \|\phi(\tau + t, x^0)\|^{m(k-1)}\, d\tau \\ &= \int_t^\infty \|\phi(\tau, x^0)\|^{m(k-1)}\, d\tau. \end{aligned}$$
Differentiating with respect to $t$ gives
$$\frac{d}{dt} V(\phi(t, x^0)) = -\|\phi(t, x^0)\|^{m(k-1)}.$$
The fourth property is proved by the following:
$$\begin{aligned} \frac{\partial V(x)}{\partial x_i} &= \int_0^\infty \frac{\partial}{\partial x_i} \|\phi(\tau, x)\|^{m(k-1)}\, d\tau \\ &= \int_0^\infty m(k-1) \|\phi(\tau, x)\|^{m(k-1)-2} \left\langle \phi(\tau, x), \frac{\partial}{\partial x_i} \phi(\tau, x) \right\rangle d\tau. \end{aligned} \tag{B.9}$$
Note that since $f \in C^1$, from [15], $\phi$ is continuously differentiable with respect to $x$. (We have used the fact $m$ is suitably large, in this step.) Note that, for fixed $a \in \mathbb{R}^n$,
$$\frac{d}{dt} \phi_i(t, a) = f_i(\phi(t, a)),$$

so writing

$$Q(t) := [q_{ij}(t)] := \left[\frac{\partial}{\partial x_j}\phi_i\right](t,a),$$

we get

$$\dot{Q}(t) = [\dot{q}_{ij}(t)] = \left[\sum_{k=1}^{n} \frac{\partial f_i(\phi(t,a))}{\partial x_k} q_{kj}(t)\right] = A(t)Q(t), \tag{B.10}$$

where $A(t) := \left[\frac{\partial f_i}{\partial x_k}\right]_{ik} (\phi(t,a))$. Note

$$\|Df(v)\| = \left\| \left[\frac{\partial f_i}{\partial x_k}\right](v) \right\| \leq K^* \|v\|^{k-1}$$

by the positive-homogeneity of order $k$ of $f(\cdot)$, see Lemma $B.2.2$. Therefore $\|Df(\phi)\| \leq K^*\|\phi\|^{k-1}$. From $(B.7)$, $\|A(t)\| \leq K^*[ar_0^{1-k} + c_1(t-t_0)]^{-1}$. With $Y = \|Q\|$, $\dot{Y}(t) \leq \|A(t)\|Y(t)$ giving

$$\|Q(t)\| = Y(t) \leq c^*(t-t_0)^s, \qquad \forall t \geq t_0$$

for some constant $c^*$, and some fixed power $s$. $\|Q(t)\|$ grows no faster than a fixed power $t^s$.

Recalling the inequality $(B.7)$, the modulus of the integral $(B.9)$ converges uniformly with respect to $x$ (with $m > s + 1 + (1/[k-1])$). Hence the integral exists, implying the bound

$$\|\nabla V(x)\| < a_3 \|x\|^{(m-1)(k-1)-1} \tag{B.11}$$

for some constant $a_3$.

We now conclude the proof by showing $\nabla V(\cdot)$ is continuous. At $x = 0$, the continuity of $\nabla V(\cdot)$ is trivial by $(B.11)$. For $x \neq 0$, let $\epsilon = \frac{1}{2}\|x\|$ and let $x_n \to x$ as $n \to \infty$. For $n \in \mathrm{N}$ define the function

$$t \mapsto f_n^i(t) := m(k-1)\|\phi(t,x_n)\|^{m(k-1)-2} \left\langle \phi(t,x_n), \frac{\partial \phi(t,x_n)}{\partial x_i} \right\rangle.$$

By continuity of $\phi$ and $\frac{\partial \phi}{\partial x_i}$,

$$f_n^i(t) \to f^i(t) := m(k-1)\|\phi(t,x)\|^{m(k-1)-2} \left\langle \phi(t,x), \frac{\partial \phi(t,x)}{\partial x_i} \right\rangle$$

as $n \to \infty$, for all $t$. For all $n$ sufficiently large, we have $\|x_n\| \leq \|x_n - x\| + \|x\| < 3\epsilon$, and so

$$\begin{aligned}\|f_n^i(t)\| &\leq m(k-1)\|\phi(t,x_n)\|^{m(k-1)-1} \left\| \frac{\partial \phi(t,x_n)}{\partial x_i} \right\| \\ &\leq g(t) := m(k-1)t^s(\|x_n\|^{1-k} + c_1 t)^{-(m-[1/(k-1)])}\end{aligned}$$

for all $t$. For $m > s+1+\frac{1}{k-1}$, $g \in L^1(\mathbb{R}_+)$ and by the Lebesgue dominated convergence theorem we may conclude that

$$\begin{aligned} \lim_{n\to\infty} \nabla V(x_n) &= \lim_{n\to\infty} \int_0^\infty f_n(t)\,dt = \int_0^\infty f(t)\,dt \\ &= \nabla V(x) \end{aligned} \tag{B.12}$$

and hence the continuity of $\nabla V(\cdot)$. This concludes the proof. □

## B.3 Inverse Liapunov theory for asymptotically stable systems

Here we state a theorem, for asymptotically stable systems, given by Kurzweil. The length of the proof of this theorem precludes its inclusion in this thesis. (Kurzweil's theorem is even more general than the one stated here.) However, the reader who is interested in Inverse Liapunov theory is encouraged to read this proof, given in [42].

**Theorem B.3.1** *Suppose $f(\cdot)$ of (B.2) is locally Lipschitz, $f(0) = 0$, and the origin is globally asymptotically stable, then there exists functions $V(\cdot)$, $U_1(\cdot)$, $U_2(\cdot)$ and $U_3(\cdot)$, such that the following conditions hold:*

- *$V(x)$ is smooth, and defined on $\mathbb{R}^n$.*
- *$U_1(\cdot)$, $U_2(\cdot)$ and $U_3(\cdot)$ are continuous on $\mathbb{R}^n$ with $U_i(0) = 0$ and $U_i(x) > 0$ elsewhere, for $i \in \{1,2,3\}$. $U_1(x) \to \infty$ as $\|x\| \to \infty$.*
- *$U_1(x) \le V(x) \le U_2(x)$ for all $x \in \mathbb{R}^n$.*
- *$\langle \nabla V(x), f(x)\rangle \le -U_3(x)$ for all $x \in \mathbb{R}^n$.*

**Proof:** See [42] for the proof of this theorem.

# Appendix C

# Uniform distributions

Here, relevant theory concerning uniformly distributed sequences modulo 1 and $2\pi$ shall be discussed. Two key references for this appendix are [41] and [25].

## C.1 Uniform distributions modulo 1

**Definition:** Let $x \in \mathbb{R}$. Define $[x] \in \mathbb{Z}$ as $[x] := \max\{z \in \mathbb{Z} \mid z \leq x\}$. Define $\{x\} := x - [x]$. $\{x\}$ is called the *fraction part* of $x$; or the *residue of $x$ modulo 1*. Note that $\{x\} \in [0, 1) =: I$.

**Definition:** Let $w = (x_n)_{n \in \mathbb{N}}$ be a sequence of real numbers. Let $N \in \mathbb{N}$ and $E \subseteq I$ be given. Define

$$A(E, N, w) := |\{x_n \in w \mid 1 \leq n \leq N \text{ and } \{x_n\} \in E\}|$$

In words, $A(E, N, w)$ is the number of the first $N$ elements, of the sequence $w$, with the fractional part in the set $E$.

**Definition:** $w = (x_n)_{n \in \mathbb{N}}$ is said to be *uniformly distributed modulo 1* if for every pair $a, b$ of real numbers with $0 \leq a < b \leq 1$ we have

$$\lim_{N \to \infty} \frac{A([a, b), N, w)}{N} = b - a$$

**Theorem C.1.1** *Let $\delta$ be a positive constant, and let $(\lambda_n)_{n \in \mathbb{N}}$ be a sequence of real numbers with $|\lambda_m - \lambda_n| \geq \delta$ for $m \neq n$. Then the sequence $(a\lambda_n)_{n \in \mathbb{N}}$ is uniformly distributed modulo 1 for almost all $a \in \mathbb{R}$.*

**Proof:** See [41, Chapter 4, Corollary 4.3]. □

**Lemma C.1.2** *If the sequence $(x_n)_{n \in \mathbb{N}}$ is uniformly distributed modulo 1, then the sequence $(x_n + \alpha)_{n \in \mathbb{N}}$, where $\alpha$ is any real number, is uniformly distributed modulo 1.*

**Proof:** See [41, Chapter 1, Lemma 1.1]. □

## C.2 Uniform distributions modulo $2\pi$

The above results relate to uniform distribution modulo 1. However for use in the setting of Chapters 2 and 6, these results are modified to cover uniform distribution modulo $2\pi$.

**Definition:** Let $x \in \mathbb{R}$. Define $[x]_{2\pi} := \max\{2\pi z \,|\, z \in \mathbb{Z} \text{ and } 2\pi z \leq x\}$. Define $\{x\}_{2\pi} := x - [x]_{2\pi}$. $\{x\}_{2\pi}$ shall be called the *residue of $x$ modulo $2\pi$*.

**Definition:** Let $w = (y_n)_{n\in\mathrm{N}}$ be a sequence of real numbers. Let $N \in \mathrm{N}$ and $E \subseteq [0, 2\pi)$ be given. Define

$$A_{2\pi}(E, N, w) := |\{y_n \in w \,|\, 1 \leq n \leq N \text{ and } \{y_n\}_{2\pi} \in E\}|\,.$$

**Definition:** The sequence $w = (y_n)_{n\in\mathrm{N}}$ of real numbers is said to be *uniformly distributed modulo* $2\pi$, if for every pair $a$, $b$ of real numbers with $0 \leq a < b \leq 2\pi$ we have

$$\lim_{N\to\infty} \frac{A_{2\pi}([a,b), N, w)}{N} = \frac{1}{2\pi}(b-a)$$

Before connecting uniformly distributed sequences modulo 1 with uniformly distributed sequences modulo $2\pi$, a following observation should be made.

**Observation:** $[2\pi x]_{2\pi} = 2\pi[x]$, and $\{2\pi x\}_{2\pi} = 2\pi\{x\}$.

This can be shown as follows. If $[x] = z \in \mathbb{Z}$ then $z \leq x$ and $z + 1 > x$. Therefore $2\pi z \in 2\pi\mathbb{Z} := \{2\pi y \,|\, y \in \mathbb{Z}\}$, $2\pi z \leq 2\pi x$, $2\pi(z+1) > 2\pi x$ and $\{2\pi x\}_{2\pi} = 2\pi x - [2\pi x]_{2\pi} = 2\pi x - 2\pi z = 2\pi\{x\}$. □

**Proposition C.2.1** *If $(x_n)_{n\in\mathrm{N}}$ is uniformly distributed modulo 1, then $(2\pi x_n)_{n\in\mathrm{N}}$ is uniformly distributed modulo $2\pi$.*

**Proof:** Let $a$, $b \in \mathbb{R}$, with $0 \leq a < b \leq 2\pi$, then

$$\begin{aligned}
A_{2\pi}([a,b), N, (2\pi x_n)_{n\in\mathrm{N}}) &= |\{2\pi x_n \in (2\pi x_n)_{n\in\mathrm{N}} \,|\, 1 \leq n \leq N \text{ and } \{2\pi x_n\}_{2\pi} \in [a,b)\}| \\
&= |\{x_n \in (x_n)_{n\in\mathrm{N}} \,|\, 1 \leq n \leq N \text{ and } 2\pi\{x_n\} \in [a,b)\}| \\
&= \left|\left\{x_n \in (x_n)_{n\in\mathrm{N}} \,|\, 1 \leq n \leq N \text{ and } \{x_n\} \in [\frac{a}{2\pi}, \frac{b}{2\pi})\right\}\right| \\
&= A\left(\left[\frac{a}{2\pi}, \frac{b}{2\pi}\right), N, (x_n)_{n\in\mathrm{N}}\right)
\end{aligned}$$

So

$$\lim_{N\to\infty} \frac{A_{2\pi}([a,b), N, (2\pi x_n)_{n\in\mathrm{N}})}{N} = \lim_{N\to\infty} \frac{A\left(\left[\frac{a}{2\pi}, \frac{b}{2\pi}\right), N, (x_n)_{n\in\mathrm{N}}\right)}{N} = \frac{1}{2\pi}(b-a) \;\square$$

**Lemma C.2.2** *If $(x_n)_{n\in\mathbb{N}}$ is uniformly distributed modulo $2\pi$, then the sequence $(x_n + \alpha)_{n\in\mathbb{N}}$ is uniformly distributed modulo $2\pi$ for any real $\alpha$.*

**Proof:** The proof is entirely analogous to the proof of Lemma $C.1.2$. □

**Proposition C.2.3** *Let $a$, $b \in \mathbb{R}$ with $0 \leq a < b \leq 2\pi$, and let $(x_n)_{n\in\mathbb{N}}$ be a sequence which is uniformly distributed modulo $2\pi$. Then there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $\{x_{n_k}\}_{2\pi} \in [a, b)$.*

**Proof:** Suppose for contradiction there are just a finite number of the $x_n$ with the property that $\{x_n\}_{2\pi} \in [a, b)$. So there exists $M \in \mathbb{N} \cup \{0\}$ with

$$A_{2\pi}([a,b), N, (x_n)_{n\in\mathbb{N}}) \leq M, \qquad \forall N \in \mathbb{N},$$

giving

$$\lim_{N\to\infty} \frac{A_{2\pi}([a,b), N, (x_n)_{n\in\mathbb{N}})}{N} \leq \lim_{N\to\infty} \frac{M}{N} = 0.$$

But since $(x_n)_{n\in\mathbb{N}}$ is uniformly distributed modulo $2\pi$,

$$\lim_{N\to\infty} \frac{A_{2\pi}([a,b), N, (x_n)_{n\in\mathbb{N}})}{N} = \frac{1}{2\pi}(b-a) > 0.$$

This is a contradiction, so the proposition is proved. □

# Appendix D

# Miscellaneous theory

## D.1 Integral estimates

The following lemma can be found in [69] and [72].

**Lemma D.1.1** *Let $\sigma(L_1) \subset \mathbb{C}_-$ and $\theta : [t_0, \omega) \to \mathbb{R}^m$ be continuous. If $w : [t_0, \omega) \to \mathbb{R}^n$ is absolutely continuous and satisfies*

$$\dot{w}(t) - L_1 w(t) - L_2\theta(t) \in \{v \in \mathbb{R}^n \mid \|v\| \leq \rho_1 + \rho_2\|\theta(t)\|\}$$

*then for $t \in [t_0, \omega)$,*

$$(i) \qquad \int_{t_0}^{t} \|\theta(s)\|\|w(s)\|\, ds \leq c_0\|w(t_0)\|^2 + c_1 \int_{t_0}^{t} \|\theta(s)\|\, ds + c_2 \int_{t_0}^{t} \|\theta(s)\|^2\, ds$$

*for some constants $c_0$, $c_1$ and $c_2$ ($c_1 = 0$ if $\rho_1 = 0$),*

$$(ii) \qquad \int_{t_0}^{t} \|\theta(s)\|\|w(s)\|\, ds \leq \mu \int_{t_0}^{t} \|\theta(s)\|(1 + \|\theta(s)\|)\, ds$$

*for some constant $\mu \geq 0$.*

**Proof:** (i)Writing $\Phi(t) = \exp(L_1 t)$, and since $\sigma(L_1) \subset \mathbb{C}_-$, there exists constants $M$, $\alpha > 0$ such that

$$\|\Phi(t)\| \leq M \exp(-\alpha t), \qquad \forall t \in \mathbb{R}$$

Since $\dot{w}(t) - L_1 w(t) - L_2\theta(t) \in \{v \in \mathbb{R}^n \mid \|v\| \leq \rho_1 + \rho_2\|\theta(t)\|\}$, we have (using [3, p.99, Lemma 1])

$$w(s) - \Phi(s - t_0)w(t_0) - \int_{t_0}^{s} \Phi(s-\tau)L_2\theta(\tau)\, d\tau \in \left\{ \int_{t_0}^{s} \Phi(s-\tau)v(\tau)\, d\tau \mid \|v(\tau)\| \leq \rho_1 + \rho_2\|\theta(\tau)\| \right\},$$

for all $s \in [t_0, \omega)$. Therefore, for all $t \in [t_0, \omega)$,

$$\begin{aligned}
\int_{t_0}^{t} \|\theta(s)\|\|w(s)\|\, ds &\le \int_{t_0}^{t} \|\theta(s)\|\|\Phi(s-t_0)\|\|w(t_0)\|\, ds \\
&\quad + \int_{t_0}^{t} \|\theta(s)\| \int_{t_0}^{s} \|\Phi(s-\tau)\| \left[(\|L_2\| + \rho_2)\|\theta(\tau)\| + \rho_1\right] d\tau\, ds \\
&\le M\|w(t_0)\| \int_{t_0}^{t} \|\theta(s)\| \exp(-\alpha(s-t_0))\, ds \\
&\quad + M(\|L_2\| + \rho_2) \int_{t_0}^{t} \|\theta(s)\| \int_{t_0}^{s} \exp(-\alpha(s-\tau))\|\theta(\tau)\|\, d\tau\, ds \\
&\quad + \rho_1 M \int_{t_0}^{t} \|\theta(s)\| \int_{t_0}^{s} \exp(-\alpha(s-\tau))\, d\tau\, ds \\
&\le M\|w(t_0)\| \left(\int_{t_0}^{t} \|\theta(s)\|^2\, ds\right)^{\frac{1}{2}} \left(\int_{t_0}^{t} \exp(-2\alpha(s-t_0))\, ds\right)^{\frac{1}{2}} \\
&\quad + M(\|L_2\| + \rho_2) \left(\int_{t_0}^{t} \|\theta(s)\|^2\, ds\right)^{\frac{1}{2}} (I(t))^{\frac{1}{2}} \\
&\quad + \rho_1 \alpha^{-1} M \int_{t_0}^{t} \|\theta(s)\| \left[1 - \exp(-\alpha(s-t_0))\right] ds
\end{aligned}$$

where

$$I(t) := \int_{t_0}^{t} \exp(-2\alpha s) \left(\int_{t_0}^{s} \exp(\alpha\tau)\|\theta(\tau)\|\, d\tau\right)^2 ds.$$

Integrating by parts and using the positivity of $\alpha$, we get

$$\begin{aligned}
I(t) &\le \alpha^{-1} \int_{t_0}^{t} \|\theta(s)\| \int_{t_0}^{s} \exp(-\alpha(s-\tau))\|\theta(\tau)\|\, d\tau\, ds \\
&\le \alpha^{-1} \left(\int_{t_0}^{t} \|\theta(s)\|^2\, ds\right)^{\frac{1}{2}} \left(\int_{t_0}^{t} \left(\int_{t_0}^{s} \exp(-\alpha(s-\tau))\|\theta(\tau)\|\, d\tau\right)^2 ds\right)^{\frac{1}{2}} \\
&= \alpha^{-1} \left(\int_{t_0}^{t} \|\theta(s)\|^2\, ds\right)^{\frac{1}{2}} (I(t))^{\frac{1}{2}}
\end{aligned}$$

whence

$$(I(t))^{\frac{1}{2}} \le \alpha^{-1} \left(\int_{t_0}^{t} \|\theta(s)\|^2\, ds\right)^{\frac{1}{2}}.$$

With $c_0 = (8\alpha)^{-\frac{1}{2}} M$, $c_1 = \rho_1 \alpha^{-1} M$ and $c_2 = c_0 + \alpha^{-1} M(\|L_2\| + \rho_2)$, it follows that for all $t \in [t_0, \omega)$,

$$\begin{aligned}
\int_{t_0}^{t} \|\theta(s)\|\|w(s)\|\, ds &\le 2c_0\|w(t_0)\| \left(\int_{t_0}^{t} \|\theta(s)\|^2\, ds\right)^{\frac{1}{2}} + c_1 \int_{t_0}^{t} \|\theta(s)\|\, ds \\
&\quad + (c_2 - c_0) \int_{t_0}^{t} \|\theta(s)\|^2\, ds \\
&\le c_0\|w(t_0)\|^2 + c_1 \int_{t_0}^{t} \|\theta(s)\|\, ds + c_2 \int_{t_0}^{t} \|\theta(s)\|^2\, ds,
\end{aligned}$$

by the difference of squares.

(ii) The proof follows the proof of (i), now noting for all $t \in [t_0, \omega)$,

$$\begin{aligned}
\int_{t_0}^{t} \|\theta(s)\| \|w(s)\| \, ds \le \int_{t_0}^{t} & \|\theta(s)\| \|\Phi(s-t_0)\| \|w(t_0)\| \, ds \\
& + \int_{t_0}^{t} \|\theta(s_1)\| \int_{t_0}^{s_1} \|\Phi(s_1 - s_2)\| \left[ (\|L_2\| + \rho_2) \|\theta(s_2)\| + \rho_1 \right] ds_2 \, ds_1 \\
\le & \; M \exp(\alpha t_0) \|w(t_0)\| \int_{t_0}^{t} \|\theta(s)\| \, ds \\
& + (\|L_2\| + \rho_2) M \int_{t_0}^{t} \|\theta(s_1)\| \int_{t_0}^{s_1} \|\theta(s_2)\| \exp(-\alpha(s_1 - s_2)) \, ds_2 \, ds_1 \\
& + M\rho_1 \int_{t_0}^{t} \|\theta(s_1)\| \int_{t_0}^{s_1} \exp(-\alpha(s_1 - s_2)) \, ds_2 \, ds_1 \\
\le & \; M \exp(\alpha t_0) \|w(t_0)\| \int_{t_0}^{t} \|\theta(s)\| \, ds \\
& + (\|L_2\| + \rho_2) M \left( \int_{t_0}^{t} \|\theta(s)\|^2 \, ds \right)^{\frac{1}{2}} (\mathcal{I}(t))^{\frac{1}{2}} \\
& - M\rho_1 \alpha^{-1} \int_{t_0}^{t} \|\theta(s)\| \left(1 - \exp(-\alpha(s - t_0))\right) ds \\
\le & \; M \exp(\alpha t_0) (\|w(t_0)\| + \alpha^{-1}\rho_1) \int_{t_0}^{t} \|\theta(s)\| \, ds \\
& + M(\|L_2\| + \rho_2) \left( \int_{t_0}^{t} \|\theta(s)\|^2 \, ds \right)^{\frac{1}{2}} (\mathcal{I}(t))^{\frac{1}{2}}
\end{aligned}$$

where

$$\mathcal{I}(t) := \int_{t_0}^{t} \exp(-2\alpha s_1) \left( \int_{t_0}^{s_1} \exp(\alpha s_2) \|\theta(s_2)\| \, ds_2 \right)^2 ds_1.$$

as before. Therefore, (ii) holds with $\mu = M(\exp(\alpha t_0)\|w(t_0)\| + \alpha^{-1}(\rho_1 + \rho_2 + \|L_2\|))$. This concludes the proof. □

## D.2 The asymptotic behaviour of a solution

In some of the analysis in this thesis, the asymptotic behaviour of absolutely continuous functions with certain properties is used. This behaviour is characterised in the following lemma, originally given in [69].

**Lemma D.2.1** *Let $I = [t_0, \infty) \subset \mathbb{R}$. If $\xi : I \to \mathbb{R}^m$ is absolutely continuous on compact subintervals of $I$ and $\xi \in L^p(I; \mathbb{R}^m)$ for some $1 \le p < \infty$ with derivative $\dot{\xi} \in L^q(I; \mathbb{R}^m)$ for some $1 \le q \le \infty$, then $\|\xi(t)\| \to 0$ as $t \to \infty$.*

**Proof:** Let $q^*$ denote the conjugate exponent of $q$, ie. $q^* = q/(q-1)$ if $1 < q < \infty$, $q^* = \infty$ if $q = 1$, and $q^* = 1$ if $q = \infty$. Write $\bar{p} = [(p/q^*) + 1]$ if $q \neq 1$, and $\bar{p} = 1$ if $q = 1$.

Seeking a contradiction, suppose $\|\xi(t)\|$ does not tend to zero as $t \to \infty$. Then there exists

$\alpha > 0$ and a sequence $(t_n)$ with $t_n \to \infty$ as $n \to \infty$, such that

$$\|\xi(t_n)\| \geq \alpha, \qquad \forall n.$$

We may assume that $t_{n+1} - t_n \geq 1/n$ for all $n$. Define a sequence $(\beta_n)$ by

$$\beta_n = \begin{cases} \alpha^{-1} \int_{t_n}^{t_n+1/n} \|\dot{\xi}(s)\|\, ds, & \text{if } q = 1, \\ \bar{p}\alpha^{-\bar{p}} \left(\int_{t_n}^{t_n+1/n} \|\xi(s)\|^p\, ds\right)^{1/q^*} \left(\int_{t_n}^{t_n+1/n} \|\dot{\xi}(s)\|^q\, ds\right)^{1/q}, & \text{if } 1 < q < \infty, \\ \bar{p}\alpha^{-\bar{p}} \|\dot{\xi}\|_\infty \int_{t_n}^{t_n+1/n} \|\xi(s)\|^p\, ds & \text{if } q = \infty. \end{cases}$$

Clearly $\beta_n \to 0$ as $n \to \infty$. Now, for almost all $t \in I$, $(\mathrm{d}/\mathrm{dt})\|\xi(t)\|^{\bar{p}} \geq -\bar{p}\|\xi(t)\|^{\bar{p}-1}\|\dot{\xi}(t)\|$. Integrating and using Holder's inequality in the case $1 < q < \infty$, it follows that for each $n \in \mathbb{N}$ and for all $t \in [t_n, t_n + 1/n]$,

$$\begin{aligned} \|\xi(t)\|^{\bar{p}} &\geq \|\xi(t_n)\|^{\bar{p}} - \begin{cases} \bar{p} \int_{t_n}^{t} \|\dot{\xi}(s)\|\, ds, & \text{if } q = 1, \\ \bar{p} \left(\int_{t_n}^{t} \|\xi(s)\|^{(\bar{p}-1)q^*}\, ds\right)^{1/q^*} \left(\int_{t_n}^{t} \|\dot{\xi}(s)\|^q\, ds\right)^{1/q}, & \text{if } 1 < q < \infty, \\ \bar{p}\|\dot{\xi}\|_\infty \int_{t_n}^{t} \|\xi(s)\|^p\, ds & \text{if } q = \infty \end{cases} \\ &\geq \alpha^{\bar{p}}(1 - \beta_n), \end{aligned}$$

which gives the following contradiction:

$$\infty > \int_T^\infty \|\xi(s)\|^p\, ds \geq \Sigma_{n=1}^\infty \int_{t_n}^{t_n+1/n} \|\xi(s)\|^p\, ds \geq \Sigma_{n=1}^\infty \frac{\alpha^p(1-\beta_n)^{p/\bar{p}}}{n}.$$

Therefore $\|\xi(t)\| \to 0$ as $t \to \infty$. □

# Bibliography


[1] Z. Artstein, *On Liapunov functions for time-varying and for controlled equations*, in Differential Equations: Qualitative Theory, S. Wagy and L. Hartvani, eds., North Holland, Amsterdam, 1986.

[2] K. J. Aström, *Theory and applications of adaptive control - a survey*, Automatica, 19 (1983), pp. 471 – 486.

[3] J. P. Aubin and A. Cellina, *Differential Inclusions*, Springer - Verlag, 1984.

[4] E. A. Barbasin and N. N. Krasovskii, *On the existence of a Ljiapunov function in the case of asymptotic stability in the large*, Prikl. Mat. Meh., 18 (1954), pp. 345–350. (Russian).

[5] R. L. Burden and J. D. Faires, *Numerical Analysis*, PWS-KENT, fourth ed., 1989.

[6] J. A. Burton and A. S. I. Zinober, *Continuous approximation of variable structure control*, International Journal of Systems Sci., 17 (1986), pp. 875–885.

[7] ——, *Multivariable continuous self-adaptive control using variable structure control system design techniques*, in The Mathematics of Control Theory, N. K. Nichols and D. H. Owens, eds., Clarendon Press, Oxford, 1992.

[8] C. I. Byrnes and A. Isidori, *Local stabilization of minimum-phase nonlinear systems*, Systems and Control Letters, 11 (1988), pp. 9–17.

[9] ——, *New results and examples in nonlinear feedback stabilization*, Systems and Control Letters, 12 (1989), pp. 437–442.

[10] ——, *Output regulation of nonlinear systems*, IEEE Transactions on Automatic Control, 35 (1990), pp. 131–140.

[11] ——, *Asymptotic stabilization of minimum phase nonlinear systems*, IEEE Transactions on Automatic Control, 36 (1991), pp. 1122–1137.

[12] ——, *Passivity, feedback equivalence, and the global stabilization of minimum phase nonlinear systems*, IEEE Transactions on Automatic Control, 36 (1991).

[13] C. I. BYRNES AND J. C. WILLEMS, *Adaptive stabilization of multivariable linear systems*, in Proc. 23rd IEEE Conf. on Decision and Control, Las Vegas, 1984, New York, pp. 1574–1577.

[14] J. B. D. CABRERA AND K. FURUTA, *Improving the robustness of nussbaum type regulators by the use of $\sigma$-modification -local results.*, Systems and Control Letters, 12 (1989), pp. 421–429.

[15] L. CESARI, *Asymptotic Behaviour and Stability Problems in Ordinary Differential Equations.*, Springer-Verlag, second ed., 1963.

[16] C. J. CHIEN AND L. C. FU, *An adaptive variable structure control for a class of nonlinear systems*, Systems and Control Letters, 21 (1993), pp. 49–57.

[17] R. F. CURTAIN AND A. J. PRITCHARD, *Functional Analysis in Modern Applied Mathematics*, vol. 132 of Mathematics in Science and Engineering, Academic Press, 1977.

[18] A. F. FILIPPOV, *Differential Equations with Discontinuous Right Hand Sides*, Kluwer Academic Publishers, Dordrecht, 1988.

[19] F. R. GANTMACHER, *The Theory of Matrices*, Chelsea Publishing Company, New York, 1977.

[20] G. H. GOLUB AND C. F. V. LOAN, *Matrix Computation*, The Johns Hopkins University Press, Baltimore and London, second ed., 1991.

[21] D. P. GOODALL, *Deterministic Feedback Stabilization of Uncertain Dynamical Systems*, PhD thesis, University of Bath, 1989.

[22] ——, *Lyapunov stabilization of a class of uncertain affine control systems*, in Variable Structure and Liapunov Control, A. S. I. Zinober, ed., Springer-Verlag, 1994.

[23] W. HAHN, *Stability of Motion*, Springer-Verlag, 1967. Translated by Arne P. Baartz.

[24] G. H. HARDY, J. E. LITTLEWOOD, AND G. POLYA, *Inequalities*, Cambridge at the University Press, 1964.

[25] G. H. HARDY AND E. M. WRIGHT, *The Theory of Numbers*, Oxford at Clarendon Press, 1975.

[26] U. HELMKE AND D. PRÄTZEL-WOLTERS, *Stability and robustness properties of universal adaptive controllers for first order linear systems*, Int. J. Control, 48 (1988), pp. 1153–1183.

[27] ——, *Stability and robustness properties of universal adaptive controllers for first order linear systems*, in Analysis and Control of Nonlinear Systems, C. I. Byrnes, C. F. Martin, and R. E. Saeks, eds., North-Holland, 1988, pp. 17–26.

[28] M. HEYMANN, J. H. LEWIS, AND G. MEYER, *Remarks on the adaptive control of linear plants with unknown high-frequency gain*, Systems and Control Letters, 5 (1985), pp. 357–362.

[29] J. Y. HUNG, W. GAO, AND J. C. HUNG, *Variable structure control: A survey*, IEEE Transactions on Industrial Electronics, 40 (1993), pp. 2–22.

[30] A. ILCHMANN, *Non-identifier-based adaptive control of dynamical systems: A survey*, IMA Journal of Mathematical Control and Information, 8 (1991), pp. 321–366.

[31] ——, *Threshold switching functions in high-gain adaptive control*, IMA Journal of Mathematics Control and Information, 8 (1991), pp. 409–429.

[32] ——, *Adaptive controllers and root-loci of multivariable minimum phase systems*, Dynamics and Control, 4 (1994), pp. 123–146.

[33] A. ILCHMANN AND H. LOGEMANN, *High-gain adaptive stabilization of multivariable linear systems - revisited*, Systems and Control Letters, 18 (1992), pp. 355–364.

[34] A. ILCHMANN AND D. H. OWENS, *Adaptive stabilization with exponential decay*, Systems and Control Letters, 14 (1990), pp. 437–443.

[35] ——, *Exponential stabilisation using non-differential gain adaption*, AMA Journal of Math. Control and Information, 7 (1991), pp. 339–349.

[36] ——, *Exponential stabilization using piecewise constant gain adaption*, in Proc. 30th IEEE Conf. on Decision and Control, Brighton, 1991, pp. 83–84.

[37] ——, *Adaptive exponential tracking for nonlinearly perturbed minimum-phase systems*, Control-Theory and Advanced Technology, 9 (1993), pp. 353–379.

[38] A. ILCHMANN AND E. P. RYAN, *Universal $\lambda$-tracking for nonlinearly-perturbed systems in the presence of noise*, Automatica, 30 (1994), pp. 337–346.

[39] A. ISIDORI, *Nonlinear Control Systems*, Springer-Verlag, second ed., 1985.

[40] H. K. KHALIL AND A. SABERI, *Adaptive stabilization of a class of nonlinear systems using high-gain feedback*, I.E.E.E. Transaction on Automatic Control, 32 (1987), pp. 1031–1035.

[41] L. KUIPERS AND H. NIEDERREITER, *Uniform Distribution of Sequences*, Pure and Applied Mathematics: A Wiley - Interscience Series of Texts, Monographs and Tracts, 1974.

[42] J. KURZWEIL, *On the Inversion of Ljapunov's Second Theorem on Stability of Motion*, vol. 24 of 2, American Mathematical Society Transactions, 1955.

[43] ——, *On the inversion of the first theorem of Ljiapunov on stability of motion*, Czech. Math. J., 5(80) (1955), pp. 382–398. (Russian).

[44] I. D. LANDAU, *Adaptive Control - The Model Reference Approach*, Marcel Dekker, New York, 1979.

[45] H. LOGEMANN, *Adaptive exponential stabilization for a class of nonlinear retarded processes*, Mathematics and Control Signal Systems, 3 (1990), pp. 255–269.

[46] H. LOGEMANN AND D. H. OWENS, *Input-output theory of high-gain adaptive stabilisation of infinite-dimensional systems with nonlinearities*, International Journal of Adaptive Control and Signal Processing, 2 (1988), pp. 193–216.

[47] I. G. MALKIN, *On the problem of inverting Ljiapunov's theorem on asymptotic stability*, Prikl. Mat. Meh., 18 (1954), pp. 129–138. (Russian).

[48] B. MÅRTENSSON, *The order of any stabilizing regulator is sufficient* a priori *information for adaptive stabilization*, Systems and Control Letters, 6 (1985), pp. 87–91.

[49] ——, *Adaptive Stabilization*, PhD thesis, Lund Institute of Technology, Sweden, 1986.

[50] ——, *Adaptive stabilization of multivariable linear systems*, Contemporary Mathematics, 68 (1987), pp. 191–225.

[51] ——, *The unmixing problem*, I.M.A. Journal of Mathematical Control and Information, 8 (1991), pp. 367–377.

[52] J. L. MASSERA, *On Liapunoff's conditions of stability*, Annals of Mathematics, 30 (1949).

[53] D. E. MILLER AND E. J. DAVISON, *A new self-tuning controller to solve the servomechanism problem*, in Proc. 26th Conf. on Decision and Control, Los Angeles, 1987, pp. 843–849.

[54] ——, *An adaptive controller which can stabilise any stabilizable and detectable LTI system*, in Analysis and Control of Nonlinear Systems, North Holland, 1988, pp. 51–58.

[55] ——, *The self-tuning robust servomechanism problem*, I.E.E.E. Transactions on Auto. Control, 34 (1989), pp. 511–523.

[56] A. S. MORSE, *Recent problems in parameter adaptive control*, in Outils et Modeles Mathematiques pour l'Automatique, l'Analyse de Systeme et le Traitement du Signal, 1983, pp. 733–740. Vol 3.

[57] ——, *An adaptive control for globally stabilizing linear systems with unknown high-frequency gains*, in Lecture Notes in Control and Information Sciences, 1984, pp. 58–68.

[58] ——, *New directions in parameter adaptive control*, in Proc. 23rd I.E.E.E. Conference on Decisions and Control, New York, 1984, pp. 1566–1568.

[59] ——, *Simple algorithms for adaptive stabilization*, in Lect. Notes in Control and Info. Sciences, vol. 105, Springer-Verlag, 1986, pp. 254–264.

[60] ——, *A $4(n+1)$-dimensional model reference adaptive stabilizer for any relative degree one or two, minimum phase system of dimension n or less*, Automatica, 23 (1987), pp. 123–125.

[61] D. R. MUDGETT AND A. S. MORSE, *Adaptive stabilization of linear systems with unknown high-frequency gains*, I.E.E.E. Transactions on Automatic Control, 30 (1985), pp. 549–554.

[62] K. S. NARENDRA, Y. H. LIN, AND L. S. VALAVANI, *Stable adaptive controller design, part ii: Proof of stability*, I.E.E.E. Transactions on Automatic Control, 25 (1980), pp. 440–448.

[63] S. NIKITIN AND S. SCHMID, *Universal Adaptive Stabilizers for One Dimensional Nonlinear Systems, Report 44*, Fachbereich Mathematik, Universitat Kaiserslautern.

[64] R. D. NUSSBAUM, *Some remarks on a conjecture in parameter adaption control*, Systems and Control Letters, 3 (1983), pp. 243–246.

[65] D. H. OWENS, D. PRÄTZEL-WOLTERS, AND A. ILCHMANN, *Positive-real structure and high-gain adaptive stabilization*, IMA J. Math. Control Info., 4 (1987), pp. 167–181.

[66] K. P. PERSIDSKII, *On a theorem of Ljiapunov*, Dokl. Akad. Nauk. SSSR, 14 (1937), pp. 541–544.

[67] E. P. RYAN, *Discontinuous feedback and universal adaptive stabilization*, in Control of Uncertain Systems, D. Hinrichsen and B. Mårtensson, eds., Birkhauser, Basel, 1990, pp. 245–258.

[68] ——, *A universal adaptive stabilizer for a class of nonlinear systems*, Systems and Control Letters, 16 (1991), pp. 209–218.

[69] ——, *Universal $W^{1,\infty}$-tracking for a class of nonlinear systems*, Systems and Control Letters, 18 (1992).

[70] ——, *Adaptive stabilization of multi-input nonlinear systems*, Robust and Nonlinear Control, 3 (1993), pp. 169–181.

[71] ——, *Adaptive stabilization and tracking control of nonlinear systems*, in Modeling Techniques for Uncertain Systems, A. B. Kurzhanshi and V. M. Veliov, eds., 1994, pp. 245–261.

[72] ——, *A nonlinear universal servomechanism*, IEEE Transactions on Automatic Control, 39 (1994), pp. 753–761.

[73] ——, *Universal controllers: Nonlinear feedback and adaption*, in Variable Structure and Liapunov Control, A. S. I. Zinober, ed., Springer-Verlag, 1994.

[74] A. SABERI AND Z. LIN, *Adaptive high-gain stabilization of minimum phase nonlinear systems*, in Proc. 29th I.E.E.E. Conf. on Decision and Control, Honolulu, Hawaii, 1990, pp. 3417–3422.

[75] A. A. STOTSKY, *Combined adaptive and variable structure control*, in Variable Structure and Liapunov Control, A. S. I. Zinober, ed., Springer-Verlag, 1994.

[76] G. STRANG, *Introduction to Applied Mathematics*, Wellesley - Cambridge Press, 1986.

[77] G. TAO AND P. A. IOANNOU, *Robust adaptive control of plants with unknown order and high frequency gain*, International Journal of Control, 53 (1991), pp. 559–578.

[78] J. M. T. THOMPSON AND H. B. STEWART, *Nonlinear Dynamics and Chaos*, Wiley, New York, 1986.

[79] S. TOWNLEY, *Simple adaptive stabilization of output feedback stabilizable distributed parameter systems*. To appear.

[80] J. H. WILKINSON, *The Algebraic Eigenvalue Problem*, Clarendon Press, Oxford, 1965.

[81] J. C. WILLEMS AND C. I. BYRNES, *Global adaptive stabilization in the absence of information on the sign of the high-frequency gain*, in Lecture Notes in Control and Infor. Sciences, Vol. 62, Sringer-Verlag, 1984, pp. 49–57.

[82] F. W. WILSON, *The structure of the level surfaces of a Lyapunov function*, Journal of Differential Equations, 3 (1967), pp. 323–329.

[83] T. YOSHIZAWA, *Stability Theory by Liapunov's Second Method*, Mathematical Society of Japan, 1966.

[84] X. J. ZHU, *A finite spectrum unmixing set for* $\mathcal{GL}(3;\mathbb{R})$, in Computation and Control, K. Bowers and J. Lund, eds., Boston: Birkhauser, 1989.

[85] A. S. I. ZINOBER, *An introduction to sliding mode variable structure control*, in Variable Structure and Liapunov Control, A. S. I. Zinober, ed., Springer-Verlag, 1994.

[86] V. I. ZUBOV, *The theory of the second method of A. M. Ljapunov*, Dokl. Akad. Nauk. SSSR, 99 (1954), pp. 341–344.

[87] ——, *Problems in the theory of Ljapunov's second method and the constriction of the general solution in the region of asymptotic stability*, Prikl. Mat. Meh., 19 (1955), pp. 179–210.